

The Linear Cost of High-Order Robust Composite Z Rotations

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Abstract

A target rotation $Z(\varphi) = e^{-i\varphi\sigma_z/2}$ of angle $0 < \varphi \leq \pi$ is synthesised from a Z evolution whose strength is wrong by an unknown factor $1 + \varepsilon$, and from exact, instantaneous, free X rotations. Only the Z evolutions consume time. We prove that N th-order robustness forces the total Z -evolution time to grow at least linearly in N , with coefficient 4:

$$T \geq 4(N+1) \left(1 - C \left(\frac{\log\left(\frac{2N+2}{\sin(\varphi/4)}\right)}{2N+2} \right)^{2/3} \right),$$

with an absolute constant C ; equivalently $c^*(\varphi) = \liminf_N T_{\min}(N, \varphi)/N \geq 4$. The angle enters the whole argument through the single number $h(0) = 1 - \cos \frac{\varphi}{2} \cos \frac{A}{2} \geq 1 - \cos \frac{\varphi}{2} = 2 \sin^2 \frac{\varphi}{4}$, the squared distance between the two endpoints of the interpolation: at $\lambda = 0$ all Z evolutions disappear and only the free X rotations survive. The remaining angles $\pi < \varphi < 2\pi$ are reduced to these by an exact reflection symmetry, $T_{\min}(N, 2\pi - \varphi) = T_{\min}(N, \varphi)$, which also identifies the two endpoints $\varphi = 0, 2\pi$, where the target $\pm I$ is the identity gate and a pure X rotation implements it at zero cost. The proof is a ladder of three arguments of increasing strength, each sharpening the same mechanism: comparing the two endpoints of the interpolation gives $c^* \geq 2/e$; passing to the scalar carrier $h = 1 - \frac{1}{2} \text{Tr}[U_1^\dagger U_\lambda]$, whose flatness order $q = 2N + 2$ is twice that of the propagator, and applying Taylor's theorem gives $c^* \geq 4/e$; and replacing Taylor's single point by the whole band gives 4. The last step rests on two estimates. Flatness is the vanishing of the first q moments, and vanishing moments let the constant term be replaced on the band by any polynomial of degree $< q$ times $e^{i\mu}$ — a replacement realised by a smoothing kernel at a cost equal to the best uniform polynomial approximation of $e^{-i\mu}$ there. That error is made exponentially small by the classical Bernstein-ellipse decay of Chebyshev coefficients whenever the bandwidth $\tau = T/2$ is below q . The threshold is exactly $\tau = q$, that is $T = 2q = 4N + 4$, so $T/N \rightarrow 4$; it is the largest coefficient this method can reach. The most demanding target is $\varphi = \pi$, where the two endpoints are maximally separated ($h(0) = 1$ for every sequence, whatever its free X angles): there the elementary bounds read $T \geq 2[\sqrt{2}(N+1)!]^{1/(N+1)}$ and $T \geq 2[(2N+2)!]^{1/(2N+2)}$, the known Low–Yoder–Chuang constructions give $T = (2N+1)\pi$, and the window is $4 \leq c^*(\pi) \leq 2\pi$, while the numerically optimal costs have increments close to 5.2 and stay between 0.84 and 0.93 of the equiangular cost up to $N = 12$. For general φ those constructions give $T = 2N\pi + \varphi$, so $4 \leq c^*(\varphi) \leq 2\pi$, and the numerical optima at $\varphi = \pi/2$, $N \leq 3$ reach 0.75–0.80 of that cost, so the upper end is not tight.

Keywords: composite pulses; robust quantum control; Chebyshev approximation; Bernstein ellipse.

1 Introduction

1.1 The model

We synthesise a target rotation

$$Z(\varphi) = e^{-i\varphi\sigma_z/2}, \quad 0 < \varphi \leq \pi, \quad (1)$$

from evolutions that are costly but imperfect,

$$Z^\varepsilon(\theta) = \exp\left[-i \frac{(1+\varepsilon)\theta}{2} \sigma_z\right], \quad \theta \geq 0, \quad (2)$$

where ε is a fixed but unknown multiplicative error of the field strength, and from rotations about the transverse axis,

$$X(\alpha) = \exp\left[-i \frac{\alpha}{2} \sigma_x\right], \quad (3)$$

assumed exact, instantaneous and free. Only the Z evolutions consume time, so the resource is the total Z angle. As ε is unknown during the synthesis it must be compensated by the algebraic structure of the sequence, not by a measurement.

A sequence of L segments

$$U(\varepsilon) = X(\alpha_L)Z^\varepsilon(\theta_L)X(\alpha_{L-1}) \cdots X(\alpha_1)Z^\varepsilon(\theta_1) \quad (4)$$

costs $T = \sum_j \theta_j$. Its robustness is read off the interpolation

$$U_\lambda = X(\alpha_L)Z(\lambda\theta_L)X(\alpha_{L-1}) \cdots X(\alpha_1)Z(\lambda\theta_1), \quad Z(\lambda\theta) = e^{-i\lambda\theta\sigma_z/2}, \quad (5)$$

which equals the physical propagator at $\lambda = 1 + \varepsilon$. The sequence is *N th-order robust* if

$$U_1 = Z(\varphi), \quad \left. \frac{d^k U_\lambda}{d\lambda^k} \right|_{\lambda=1} = 0, \quad k = 1, \dots, N, \quad (6)$$

so that $U(\varepsilon) = Z(\varphi) + O(\varepsilon^{N+1})$. Let $T_{\min}(N, \varphi)$ be the infimum of T over such sequences and

$$c^*(\varphi) := \liminf_{N \rightarrow \infty} \frac{T_{\min}(N, \varphi)}{N}. \quad (7)$$

The question is how fast $T_{\min}(N, \varphi)$ must grow with the robustness order N : can arbitrarily many free, exact X rotations reduce the cost of high-order robustification to $O(1)$, or to any sublinear function of N ? Since every Z evolution is imperfect in the same way, the answer is not obvious, and the model is the idealised limit of an experimentally natural situation. In Rydberg atom arrays, for instance, a controlled phase is generated by free evolution under a distance-dependent interaction whose strength V is known only to within a few percent, while the Rydberg pulses that rotate the qubit are fast and precise [6]; there the imperfect Z evolution of (2) is the free evolution and the exact X rotations of (3) are the Rydberg pulses.

We prove everything for $0 < \varphi \leq \pi$. This is no restriction on the problem: the reflection $\varphi \mapsto 2\pi - \varphi$ is an exact symmetry of the model, $T_{\min}(N, 2\pi - \varphi) = T_{\min}(N, \varphi)$ (Proposition 2), so the remaining angles follow by relabelling. The symmetry is also what makes the two endpoints behave alike; it is proved in Section 6.

Sequences of this kind generalise the composite pulses of nuclear magnetic resonance [3, 1]. In the standard composite-pulse model every elementary rotation is both imperfect and costly, and lower bounds on the operation time there go back to Kukita, Kiya and Kondo [2], who proved the isoperimetric bound $L > \sqrt{\theta(4\pi - \theta)}$ for first-order robustness against the pulse-length error; because that model also charges for the axis-changing rotations, its bounds do not transfer to T . In the model (4) the axis changes are free and the resource T is the quantity bounded here.

1.2 Results

Our main result is that robustness forces a *linear* cost with coefficient at least 4.

Theorem 1. *There are absolute constants C and N_0 such that for every $N \geq N_0$ and every target angle $0 < \varphi \leq \pi$,*

$$T_{\min}(N, \varphi) \geq 4(N+1) \left(1 - C \left(\frac{\log\left(\frac{2N+2}{\sin(\varphi/4)}\right)}{2N+2} \right)^{2/3} \right), \quad (8)$$

and consequently

$$c^*(\varphi) = \liminf_{N \rightarrow \infty} \frac{T_{\min}(N, \varphi)}{N} \geq 4. \quad (9)$$

The dependence on the target angle enters only through the single number $\sin(\varphi/4)$: for $0 < \varphi \leq \pi$,

$$h(0) = 1 - \cos \frac{\varphi}{2} \cos \frac{A}{2} \geq 1 - \cos \frac{\varphi}{2} = 2 \sin^2 \frac{\varphi}{4}, \quad (10)$$

as shown in Corollary 9, and this is the only place where φ appears anywhere in the argument. The quantity $h(0) = \frac{1}{2} \|U_1 - U_0\|^2$ is the squared operator-norm distance between the two endpoints of the interpolation. No uniformity in φ is claimed: for each fixed φ the factor in (8) tends to 1 as $N \rightarrow \infty$, so the linear coefficient is 4 for every target angle.

The remaining angles are covered by symmetry.

Proposition 2 (Reflection). *For every $N \geq 0$ and every real φ ,*

$$T_{\min}(N, 2\pi - \varphi) = T_{\min}(N, \varphi). \quad (11)$$

Corollary 3. *For every $0 < \varphi < 2\pi$ the conclusions of Theorem 1 hold with $\sin \frac{\varphi}{4}$ replaced by $s(\varphi) := \min\{\sin \frac{\varphi}{4}, \cos \frac{\varphi}{4}\}$, because $\sin \frac{2\pi - \varphi}{4} = \cos \frac{\varphi}{4}$.*

At the two endpoints the target is $\pm I$, which is the identity gate up to a global phase, and a pure X rotation implements it exactly at zero cost:

$$T_{\min}(N, 0) = T_{\min}(N, 2\pi) = 0 \quad \text{for every } N. \quad (12)$$

By Proposition 2 the two statements are in fact the same one, since $2\pi - 0 = 2\pi$; the range of the problem is therefore exactly the open interval, as Remark 10 explains.

The two intermediate results of the ladder are worth stating, because they isolate the mechanism, and each is proved in a few lines.

Proposition 4 (Endpoint comparison). *Every N th-order robust sequence implementing $Z(\varphi)$, $0 < \varphi \leq \pi$, satisfies*

$$T \geq 2[2(N+1)! \sin(\varphi/4)]^{1/(N+1)}, \quad (13)$$

and consequently $c^*(\varphi) \geq 2/e \approx 0.7358$.

Proposition 5 (Doubled flatness order). *Every N th-order robust sequence implementing $Z(\varphi)$, $0 < \varphi \leq \pi$, satisfies*

$$T \geq 2[2(2N+2)! \sin^2(\varphi/4)]^{1/(2N+2)}, \quad (14)$$

and consequently $c^*(\varphi) \geq 4/e \approx 1.4715$.

The coefficient 4 in Theorem 1 is the ceiling of the method, not of the problem: at $\tau = q$ the moments (22) cease to force a contradiction (Section 7). For $\varphi = \pi$ the constructions of Low, Yoder and Chuang [4, 5] give $T = 2N\pi + \varphi$, so that

$$4 \leq c^*(\varphi) \leq 2\pi \quad (0 < \varphi < 2\pi), \quad (15)$$

the upper bound for the remaining angles following from the symmetry; and Section 7 collects the evidence that the upper end is not tight.

1.3 The plan of the proof

Write

$$\tau := \frac{T}{2}, \quad q := 2N + 2, \quad h(\lambda) := 1 - \frac{1}{2} \text{Tr}[U_1^\dagger U_\lambda], \quad (16)$$

and call h the *error carrier*. Four elementary properties of h , collected in Proposition 7, drive everything: h is a finite sum of oscillations of bandwidth τ ; it takes values in $[0, 2]$; it is flat of order q at $\lambda = 1$; and $h(0) \geq 2 \sin^2(\varphi/4) > 0$.

The proof is a ladder of three arguments, each using the same four properties and each extracting more from them. Sections 3–5 are these three steps; the rest of the paper discusses the target angle, the sharpness of the result and the known constructions, and can be read independently of the details of the proofs.

- L1** *Compare the two endpoints.* At $\lambda = 0$ all Z evolutions disappear and $U_0 = X(A)$ is a pure X rotation; at $\lambda = 1$ the propagator equals $Z(\varphi)$ by definition. The first N derivatives vanish at $\lambda = 1$, so this separation can only be produced by the $(N + 1)$ st derivative, whose size is bounded by τ^{N+1} . This costs nothing but Taylor's theorem and gives $T \gtrsim \frac{2}{e}N$ (Section 3).
- L2** *Pass to the carrier.* The same argument applied to h instead of to U_λ is twice as strong, because h is a *squared* quantity and therefore flat to order $q = 2N + 2$ rather than $N + 1$. The factor 2 in the flatness order is worth a factor 2 in the coefficient, and gives $T \gtrsim \frac{4}{e}N$ (Section 4).
- L3** *Use the whole band.* Both arguments above inspect the carrier at a single point, the unknown point of the Taylor remainder. Flatness is far more than that: it says that the first q moments of the carrier vanish, and hence that on the band $|\mu| \leq \tau$ the constant 1 may be replaced by any polynomial of degree $< q$ times $e^{i\mu}$. Averaging against a smoothing kernel realises that replacement at a cost equal to the best approximation error of $e^{-i\mu}$ on the band, and the Bernstein ellipse makes that error exponentially small in q as soon as $\tau < q$. Optimising the bandwidth then gives $T \geq 4N(1 - o(1))$ (Section 5).

Each of the three steps is self-contained except for the two estimates of Section 5.2, which are stated there together with the idea of their proof; the proofs themselves, being computations rather than ideas, are deferred to Appendices A and B.

1.4 Relation to previous work

Low, Yoder and Chuang established an $L = \Omega(N)$ lower bound on the number of primitive pulses in the composite-pulse model and constructed $L = 2N$ sequences, thereby proving optimal linear pulse-count scaling [4]. Their elementary rotations are themselves imperfect and costly, whereas the present model assigns cost only to the imperfect Z evolution and treats exact X rotations as free; hence an L lower bound does not by itself imply a lower bound on our time cost T , and conversely our bound on T does not follow from theirs. The constructions of [4, 5], on the other hand, do transfer to the present model and supply the upper bound in (15) (Section 7).

Kukita, Kiya and Kondo derived a geometric lower bound on the operation time for first-order pulse-length-error-robust composite gates [2]. Their model assigns time cost to every elementary rotation, including the rotations used to change the control axis; in the present model those axis-changing X rotations are instantaneous, so their geometric length bound cannot be transferred to T . The present theorem instead exploits the analytic dependence on the common scale parameter λ and remains valid for arbitrary robustness order.

1.5 Organisation

Section 2 fixes the model and proves the four properties of the carrier h together with the endpoint identity (10). Sections 3, 4 and 5 are the three steps of the ladder, culminating in Theorem 1; the two

estimates they use are stated in Section 5.2 and proved in Appendices A and B. Section 6 treats the target angle: the reflection symmetry, the two endpoints and the case $\varphi = \pi$. Section 7 discusses how sharp the bounds are — the exact threshold of the method and the known constructions — and Section 8 states the open problems. Throughout, $\|M\|$ denotes the operator norm of a matrix M ; $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. All constants are absolute and no attempt is made to optimise them.

2 The model and the error carrier

Model 6. A sequence is a pair of real vectors (θ, α) with $\theta = (\theta_1, \dots, \theta_L)$, $\theta_j \geq 0$, and $\alpha = (\alpha_1, \dots, \alpha_L)$. Its cost is $T(\theta) = \sum_{j=1}^L \theta_j$, and its interpolation is (5), so that the physical propagator (4) is $U_{1+\varepsilon}$. The sequence is admissible of order N for the angle φ if (6) holds.

Proposition 7 (The carrier). *Let U_λ be admissible of order N for φ with cost T , and let $h(\lambda) = 1 - \frac{1}{2} \text{Tr}[U_1^\dagger U_\lambda]$. Then*

- (i) $h(\lambda) = \sum_m c_m e^{i\mu_m \lambda}$ is a finite sum with $|\mu_m| \leq \tau = T/2$;
- (ii) $0 \leq h \leq 2$ on $\mathbb{R}$;
- (iii) $h^{(k)}(1) = 0$ for $k = 0, 1, \dots, q-1$, where $q = 2N+2$;
- (iv) $h(0) \geq 2 \sin^2(\varphi/4)$ for $0 < \varphi \leq \pi$.

Proof. (i) Since σ_z has eigenvalues ± 1 , $Z(\lambda\theta) = e^{-i\lambda\theta/2 \frac{I+\sigma_z}{2}} + e^{+i\lambda\theta/2 \frac{I-\sigma_z}{2}}$. Expanding the product (5) gives finitely many exponentials with frequencies $\mu = \frac{1}{2} \sum_j \varepsilon_j \theta_j$, $\varepsilon_j = \pm 1$, and $|\mu| \leq \frac{1}{2} \sum_j \theta_j = \tau$.

(ii) For $V = U_1^\dagger U_\lambda$ unitary we have $h = 1 - \frac{1}{2} \text{Tr} V$ and $|\frac{1}{2} \text{Tr} V| \leq 1$, so $0 \leq h \leq 2$.

(iii) Write $V = aI + i\mathbf{b} \cdot \boldsymbol{\sigma}$ with $a^2 + |\mathbf{b}|^2 = 1$. By (6) the derivatives of V of orders $1, \dots, N$ vanish at $\lambda = 1$; hence those of a and of $\mathbf{b}$ do, while $a(1) = 1$ and $\mathbf{b}(1) = 0$. Thus $\mathbf{b}$ vanishes to order at least $N+1$ there. Since $h = 1 - a = \frac{1-a^2}{1+a} = \frac{|\mathbf{b}|^2}{1+a}$ in a neighbourhood of $\lambda = 1$, where $1+a = 2 \neq 0$, and since the numerator vanishes to order at least $2N+2$ there, so does h .

(iv) This is Corollary 9 below, whose proof is independent of the other items. $\square$

The doubling in (iii) deserves emphasis: one order of vanishing of $V - I$ produces *two* orders of vanishing of h , because h is quadratic in $\mathbf{b}$. This is the whole content of the second rung of the ladder.

Proposition 8 (Endpoint separation). *Let $U, V \in SU(2)$. Then*

$$\|U - V\|^2 = 2 - \text{Tr}(U^\dagger V). \quad (17)$$

Moreover, for every real A and every $0 < \varphi \leq \pi$,

$$\|X(A) - Z(\varphi)\|^2 = 2 - 2 \cos \frac{A}{2} \cos \frac{\varphi}{2} \geq 2 - 2 \cos \frac{\varphi}{2} = 4 \sin^2 \frac{\varphi}{4}. \quad (18)$$

Proof. For (17) put $W = U^\dagger V$, which lies in $SU(2)$ and therefore has eigenvalues $e^{\mp i\gamma/2}$ for some $\gamma \in [0, 2\pi]$. As $U - V = U(I - W)$ with U unitary, $\|U - V\| = \|I - W\|$, and $I - W$ is normal with eigenvalues $1 - e^{\mp i\gamma/2}$, so $\|U - V\| = \max_\pm |1 - e^{\mp i\gamma/2}| = 2|\sin(\gamma/4)|$ and $\|U - V\|^2 = 4 \sin^2(\gamma/4) = 2 - 2 \cos(\gamma/2) = 2 - \text{Tr} W$.

For (18) we compute the trace: writing $Z(\varphi) = c_{\varphi/2} I - i s_{\varphi/2} \sigma_z$ and $X(-A) = c_{A/2} I + i s_{A/2} \sigma_x$ with $c_x = \cos x$, $s_x = \sin x$, and using $\text{Tr} \sigma_i = 0$ and $\text{Tr}(\sigma_x \sigma_z) = 0$, we get $\text{Tr}[X(A)^\dagger Z(\varphi)] = \text{Tr}[X(-A)Z(\varphi)] = 2 \cos \frac{A}{2} \cos \frac{\varphi}{2}$. Inserting this into (17) and minimising over A gives $\min_A \|X(A) - Z(\varphi)\|^2 = 2 - 2|\cos \frac{\varphi}{2}|$, and for $0 < \varphi \leq \pi$ we have $\cos \frac{\varphi}{2} \geq 0$, so $|\cos \frac{\varphi}{2}| = \cos \frac{\varphi}{2}$ and the minimum is $2 - 2 \cos \frac{\varphi}{2} = 4 \sin^2 \frac{\varphi}{4}$. $\square$

Corollary 9. Let U_λ be admissible of order N for φ with $0 < \varphi \leq \pi$, and let $A = \sum_{j=1}^L \alpha_j$ be the total X angle. Then

$$h(0) = 1 - \cos \frac{\varphi}{2} \cos \frac{A}{2} \geq 1 - \cos \frac{\varphi}{2} = 2 \sin^2 \frac{\varphi}{4} > 0, \quad (19)$$

and equivalently $h(0) = \frac{1}{2} \|U_1 - U_0\|^2$, the squared operator-norm distance between the two endpoints of the interpolation.

Proof. At $\lambda = 0$ every Z factor is the identity, so $U_0 = X(\alpha_L) \cdots X(\alpha_1) = X(A)$. Hence $\text{Tr}[U_1^\dagger U_0] = \text{Tr}[Z(\varphi)^\dagger X(A)] = 2 \cos \frac{\varphi}{2} \cos \frac{A}{2}$ by the computation in the proof of Proposition 8, and (19) follows by minimising over the free total angle A , using $\cos \frac{\varphi}{2} \geq 0$ for $\varphi \leq \pi$ and $1 - \cos \frac{\varphi}{2} = 2 \sin^2 \frac{\varphi}{4}$. For the equivalent form, apply (17) with $U = U_1$ and $V = U_0$: $\|U_1 - U_0\|^2 = 2 - \text{Tr}[U_1^\dagger U_0] = 2h(0)$. $\square$

Remark 10 (The two endpoint angles are degenerate). At $\varphi = 0$ the target is I , and the sequence with a single segment of angle $\theta_1 = 0$ and $\alpha_1 = 0$ has $U_\lambda \equiv I$ for all λ : it is exactly robust to every order at cost $T = 0$. At $\varphi = 2\pi$ the target is $-I$, and the sequence with a single segment of angle $\theta_1 = 0$ and $\alpha_1 = 2\pi$ has $U_\lambda \equiv X(2\pi) = -I$ for all λ , again exactly robust to every order at cost $T = 0$. Both targets are the identity gate up to a global phase, and therefore require no Z evolution at all; as $T \geq 0$ always, both infima vanish: (12). This is also where $\sin(\varphi/4)$ vanishes, so the endpoint bound (19) is vacuous there and no positive lower bound can hold. The range $(0, 2\pi)$ is therefore exactly the set of angles at which the question is non-trivial, and by Proposition 2 the two endpoints are mirror images of one another.

Remark 11 (Conventions). Both the Z evolution and the X control are generated by positive Pauli generators, $Z^\varepsilon(\theta) = e^{-iH_Z t}$ with $H_Z = \frac{\omega}{2}(1 + \varepsilon)\sigma_z$ and $t = \theta/\omega$, and $X(\alpha) = e^{-iH_X t'}$ with $H_X = \frac{\alpha}{2t'}\sigma_x$. All statements are invariant under $\sigma_z \rightarrow -\sigma_z$, so the sign in the exponent of Z is immaterial; only the relative sign between the target (1) and the free control (3) matters, and it is fixed by (5). The norm bounds below and the identity (17) both hold for the operator norm, which is what $\|\cdot\|$ denotes throughout.

3 First bound: comparing the two endpoints ($c^* \geq 2/e$)

The first rung uses nothing but Taylor's theorem and the separation of the two endpoints of the interpolation. Two estimates are needed; both are elementary, and their proofs are in Appendix B.

Lemma 12 (Derivative bound). For every $\lambda \in \mathbb{R}$ and every $k \geq 0$,

$$\|U_\lambda^{(k)}\| \leq \left(\frac{T}{2}\right)^k = \tau^k, \quad \text{and} \quad |h^{(k)}| \leq \tau^k. \quad (20)$$

The idea is that $\frac{d^m}{d\lambda^m} Z(\lambda\theta_j)$ has norm $(\theta_j/2)^m$, so the product rule applied to (5) yields a multinomial sum whose value is $(\sum_j \theta_j/2)^k = \tau^k$; the second statement follows from $|\text{Tr } M| \leq 2\|M\|$ for a 2×2 matrix. We use the lemma only in this form, and only for $k \leq q$.

Lemma 13 (Taylor remainder). If U_λ satisfies the robustness conditions (6), then

$$U_0 - U_1 = \frac{(-1)^{N+1}}{N!} \int_0^1 t^N U_t^{(N+1)} dt, \quad \text{so} \quad \|U_0 - U_1\| \leq \frac{\tau^{N+1}}{(N+1)!}. \quad (21)$$

Here the idea is simply to expand U_λ about $\lambda = 1$ with the integral remainder and to evaluate at $\lambda = 0$: the terms of order $1, \dots, N$ die by (6), and what survives is one integral of $U^{(N+1)}$, which Lemma 12 bounds by τ^{N+1} .

Proof of Proposition 4. Let the sequence be admissible of order N with cost T . By Corollary 9, $U_0 = X(A)$ and $U_1 = Z(\varphi)$, so Proposition 8 gives $\|U_0 - U_1\| \geq 2 \sin \frac{\varphi}{4}$, while Lemmas 13 and 12 give $\|U_0 - U_1\| \leq \tau^{N+1}/(N+1)!$. Hence

$$2 \sin \frac{\varphi}{4} \leq \frac{\tau^{N+1}}{(N+1)!}, \quad \text{that is} \quad T = 2\tau \geq 2[2(N+1)! \sin(\varphi/4)]^{1/(N+1)},$$

which is (13). By Stirling's formula, $[(N+1)!]^{1/(N+1)} = \frac{N+1}{e} (1 + O(\frac{\log N}{N}))$, so for fixed φ , $T \geq \frac{2(N+1)}{e} (2 \sin \frac{\varphi}{4})^{1/(N+1)} (1 + o(1))$ with $(2 \sin \frac{\varphi}{4})^{1/(N+1)} \rightarrow 1$; dividing by N and letting $N \rightarrow \infty$ gives $c^*(\varphi) \geq 2/e$. $\square$

Remark 14 (What limits this rung). The bound (21) is as large as the derivative bound permits, so no refinement of this step can do better: the only information about the sequence that the argument uses is the size of the $(N+1)$ st derivative at one unknown point. Two things are wasted. The propagator U_λ is flat only to order $N+1$, whereas its scalar carrier is flat to twice that order (Section 4). And flatness is a property of the whole band $|\mu| \leq \tau$ on which h lives, whereas Taylor's theorem samples it at a single point (Section 5).

4 Second bound: doubling the flatness order ($c^* \geq 4/e$)

The second rung replaces the propagator by the carrier, which by Proposition 7(iii) is flat to order $q = 2N + 2$, and then repeats the argument of Section 3 verbatim.

Proof of Proposition 5. Let the sequence be admissible of order N with cost T . By Proposition 7(iii), $h^{(k)}(1) = 0$ for $k = 0, \dots, q-1$ with $q = 2N + 2$, so Taylor's theorem of order q about $\lambda = 1$ gives a point $\xi \in (0, 1)$ with

$$h(0) = \frac{(-1)^q h^{(q)}(\xi)}{q!}, \quad \text{whence} \quad |h(0)| \leq \frac{\tau^q}{q!}$$

by Lemma 12. With $h(0) \geq 2 \sin^2 \frac{\varphi}{4}$ from Proposition 7(iv) this yields $\tau^q \geq 2 q! \sin^2 \frac{\varphi}{4}$, that is $T = 2\tau \geq 2[2 q! \sin^2 \frac{\varphi}{4}]^{1/q}$, which is (14) in view of $q = 2N + 2$. Dividing by N and using $(q!)^{1/q} = \frac{q}{e} (1 + o(1))$ gives $T \geq \frac{2q}{e} (1 + o(1)) = \frac{4(N+1)}{e} (1 + o(1))$ and hence $c^*(\varphi) \geq 4/e$. $\square$

The improvement over Section 3 is exactly the factor 2 in $q = 2(N+1)$, and it costs nothing: the same three ingredients — a derivative bound, Taylor's theorem, and the endpoint separation — are used again, only with U_λ replaced by $h(\lambda)$.

Remark 15 (What limits this rung). The Taylor remainder $\tau^q/q!$ is again evaluated at a single unknown point ξ . The gain from $\tau^q/q!$ to an *exponentially* small quantity requires using the flatness on the whole band, which is what the third rung does.

5 Third bound: using the whole band ($c^* \geq 4$)

We now use the carrier on the whole band. The section has three parts: the reformulation of flatness as a set of moment conditions, which is the idea of the method; the two estimates about one oscillation on a band; and the assembly.

5.1 Flatness is a set of moments

Because h is a finite exponential sum, its flatness at $\lambda = 1$ is a statement about its coefficients, and it is exactly the statement that a finite set of moments vanishes.

Lemma 16 (Flatness = vanishing moments). *Let $h(\lambda) = \sum_m c_m e^{i\mu_m \lambda}$ be a finite sum and let $q \geq 1$. Then $h^{(k)}(1) = 0$ for $k = 0, 1, \dots, q-1$ if and only if*

$$\sum_m c_m \mu_m^k e^{i\mu_m} = 0, \quad k = 0, 1, \dots, q-1. \quad (22)$$

In that case, for every polynomial P of degree $< q$,

$$h(0) = \sum_m c_m = \sum_m c_m [1 - P(\mu_m) e^{i\mu_m}], \quad |1 - P(\mu) e^{i\mu}| = |e^{-i\mu} - P(\mu)|. \quad (23)$$

Proof. $h^{(k)}(\lambda) = \sum_m c_m (i\mu_m)^k e^{i\mu_m \lambda}$, so $h^{(k)}(1) = 0$ is $\sum_m c_m (i\mu_m)^k e^{i\mu_m} = 0$; multiplying by $(-i)^k$ gives (22), and the equivalence is immediate. If $P(\mu) = \sum_{k < q} p_k \mu^k$ then $\sum_m c_m P(\mu_m) e^{i\mu_m} = \sum_{k < q} p_k \sum_m c_m \mu_m^k e^{i\mu_m} = 0$ by (22), which is the first identity in (23); the second is $|1 - P(\mu) e^{i\mu}| = |e^{-i\mu}| \cdot |e^{-i\mu} - P(\mu)|$. $\square$

The content of (23) is a trade. On the one hand $h(0)$ is the sum of the weights c_m ; on the other hand, because the weights annihilate polynomials of degree $< q$, that sum is unchanged if the constant 1 is replaced by $P(\mu) e^{i\mu}$. So a quantity we cannot compute may be replaced by one we can, at the price of the error $|e^{-i\mu} - P(\mu)|$ on the band. Everything that follows consists in choosing P and paying that price.

5.2 Two estimates about one oscillation on a band

Realising the replacement (23) by averaging means choosing an integrable kernel κ with $\hat{\kappa}(\mu) = 1 - P(\mu) e^{i\mu}$ on the band. Then, by Proposition 7(i),

$$\int_{\mathbb{R}} h(t) \kappa(t) dt = \sum_m c_m \hat{\kappa}(\mu_m) = h(0), \quad \hat{\kappa}(\mu) := \int_{\mathbb{R}} \kappa(t) e^{i\mu t} dt, \quad (24)$$

so $|h(0)| \leq \|\kappa\|_{L^1} \|h\|_{\infty} \leq 2\|\kappa\|_{L^1}$. The cost of the replacement is therefore the L^1 norm of κ , and that is governed by the approximation error of $e^{-i\mu}$ on the band.

Lemma 17 (Smoothing). *Let h satisfy Proposition 7(i),(ii) and the moment conditions (22), let P be a polynomial of degree $< q$, and put $\eta := \sup_{|\mu| \leq \tau} |e^{-i\mu} - P(\mu)|$. If $1 \leq \tau \leq q$, $q \geq 2$ and $\eta \leq 1$, then*

$$|h(0)| \leq c_0 q^2 \sqrt{\eta}, \quad (25)$$

where c_0 is an absolute constant (the proof gives $c_0 = 45$).

The construction is one line: multiply $\Psi(\mu) := 1 - P(\mu) e^{i\mu}$ by a smooth cut-off χ equal to one on $[-\tau, \tau]$ and vanishing just outside, and invert the Fourier transform. Since $\hat{\kappa} = \Psi\chi$ is smooth and compactly supported, κ decays like t^{-2} , and two integrations by parts bound $\|\kappa\|_{L^1}$ by the geometric mean of $\int |\hat{\kappa}|$ and $\int |\hat{\kappa}''|$; the first is $O(\tau\eta)$ because $\hat{\kappa} = \Psi$ on the band, and the second is $O(q^4/\tau)$ because Markov's inequality bounds the derivatives of the polynomial P on the band. The product is $O(q^4\eta)$, whose square root is the $q^2\sqrt{\eta}$ of (25). Only the *polynomial* nature of the factor q^2 matters below, because it will be annihilated by the exponential decay of the next lemma. The proof is in Appendix A; the hypotheses $\tau \geq 1$ and $\tau \leq q$ are met in the application below.

Lemma 18 (Ellipse decay). *Let $\tau > 0$, $q \geq 1$ and $\rho > 1$. Then some polynomial P of degree $< q$ satisfies*

$$\sup_{|\mu| \leq \tau} |e^{-i\mu} - P(\mu)| \leq \frac{2}{1 - \rho^{-1}} \exp\left(\frac{\tau}{2}(\rho - \rho^{-1})\right) \rho^{-q}. \quad (26)$$

If $\tau = \rho_0 q$ with $0 < \rho_0 < 1$ and $\rho = \rho(\rho_0) := (1 + \sqrt{1 - \rho_0^2})/\rho_0$, then the right-hand side of (26) equals $(1 - \rho^{-1})^{-1} 2e^{qg(\rho_0)}$, where

$$g(z) := \sqrt{1 - z^2} - \operatorname{arccosh} \frac{1}{z} = \sin \theta - \operatorname{artanh}(\sin \theta) \Big|_{z=\cos \theta} \quad (27)$$

satisfies $g < 0$ and $g' > 0$ on $(0, 1)$, $g(1) = 0$, and $g(z) = -\frac{2\sqrt{2}}{3}(1 - z)^{3/2} + O((1 - z)^{5/2})$ as $z \uparrow 1$.

The proof is the classical Bernstein-ellipse argument, and it has three ingredients, all standard. First, rescale the band to $[-1, 1]$ and expand $e^{-i\tau u}$ in Chebyshev polynomials; if the function is bounded by M on the Bernstein ellipse $\mathcal{E}_\rho$ with foci ± 1 and semi-axis sum ρ , then its Chebyshev coefficients decay as ρ^{-k} , so truncating at $k = q - 1$ leaves an error $O(M\rho^{-q})$. Second, $M \leq \exp(\frac{\tau}{2}(\rho - \rho^{-1}))$, because that is how far the ellipse reaches in the imaginary direction. Third, and this is the only choice to be made,

ρ is optimised: the exponent $\frac{\tau}{2}(\rho - \rho^{-1}) - q \ln \rho$ is stationary at $\rho = (1 + w)/\rho_0$ with $w = \sqrt{1 - \rho_0^2}$ when $\tau = \rho_0 q$, where it takes the value $qg(\rho_0)$. The function g vanishes at $z = 1$ — that is the threshold discussed in Section 7 — and its two leading terms cancel there, leaving $g = O((1 - z)^{3/2})$. The details, including the minimisation and the properties of g , are in Appendix B.

5.3 The assembly

We can now finish. The two estimates say that the approximation error η can be made exponentially small by taking the bandwidth τ below the flatness order q , and that a polynomial factor q^2 has to be paid for it. The proof is a contradiction: we assume that τ is a fixed fraction of q and let q grow.

Proof of Theorem 1. Fix $0 < \varphi \leq \pi$, put $s := \sin(\varphi/4) > 0$ and $q := 2N + 2 \geq 2$, and let an admissible sequence of order N have cost T and $\tau = T/2$. We show that τ cannot be small.

Suppose, to the contrary, that

$$\tau \leq \rho_0 q, \quad \rho_0 := 1 - \left(\frac{A \log \frac{q}{s}}{q} \right)^{2/3}, \quad (28)$$

with $A \geq 1$ to be chosen. (If $\rho_0 \leq 0$ this is impossible and there is nothing to prove; if $\rho_0 q < 1$ then $T = 2\tau < 2$, which is already smaller than the right-hand side of (8) for large N ; and if $\tau < 1$ we may replace τ by $\max\{\tau, 1\} \leq \rho_0 q$, all the estimates below being monotone in the bandwidth. So we may assume $0 < \rho_0 < 1$ and $1 \leq \tau \leq q$.) Write $L := \log \frac{q}{s}$, so that $1 - \rho_0 = (AL/q)^{2/3}$.

Step 1: a good polynomial. Apply Lemma 18 with $\tau = \rho_0 q$ and $\rho = \rho(\rho_0)$. Since $g \leq -\frac{1}{3}(1 - z)^{3/2}$ (Appendix B) and $1 - \rho_0 = (AL/q)^{2/3}$, the exponent is at most $-\frac{AL}{3}$, while the prefactor $2/(1 - \rho^{-1})$ is at most $4(q/AL)^{1/3}$; hence Lemma 18 provides a polynomial P of degree $< q$ with

$$\eta \leq 4 \left(\frac{q}{AL} \right)^{1/3} e^{-AL/3}, \quad (29)$$

which is ≤ 1 as soon as $q \geq 2$ and $A \geq 1$.

Step 2: the two estimates contradict the endpoint bound. By Lemma 16 the hypothesis of Lemma 17 is met with this P , so (25) and Proposition 7(iv) give

$$2s^2 \leq h(0) \leq c_0 q^2 \sqrt{\eta} \leq 2c_0 (q/s)^{-A/6} q^{13/6}. \quad (30)$$

The last expression is the point of the whole computation: it contains $q^{13/6} \cdot q^{-A/6} = q^{(13-A)/6}$, an exponential gain in q provided $A > 13$, multiplied by the algebraic factor $s^{A/6}$ that the choice (28) produces. Rearranging (30) with $A = 14$ gives

$$q^{1/6} (A \log \frac{q}{s})^{1/6} s^{-1/3} \leq c_0, \quad (31)$$

whose left-hand side is at least $q^{1/6} (A \log q)^{1/6}$, since $s^{-1/3} \geq 1$ and $q/s \geq q$. Hence (31) can hold only if

$$A q \log q \leq c_0^6. \quad (32)$$

Since $q \mapsto q \log q$ is increasing on $[1, \infty)$, (32) fails for every $q > q_0$, where q_0 is the absolute constant solving $A q_0 \log q_0 = c_0^6$ — numerically 3.42×10^7 for the values $A = 14$, $c_0 = 45$ produced by the two estimates. So no admissible sequence of order N with $q = 2N + 2 > q_0$ satisfies (28), and for all such N

$$T = 2\tau > 2\rho_0 q = 4(N + 1) \left(1 - \left(\frac{A \log \frac{2N+2}{s}}{2N+2} \right)^{2/3} \right), \quad (33)$$

which is (8) with $C = A = 14$. Dividing by N and letting $N \rightarrow \infty$ gives (9), and the endpoints are Remark 10. $\square$

Remark 19 (The two steps in one line each). A reader who wants to remember the mechanism rather than the constants should retain this. Lemma 18 says that $\eta \leq e^{qg(\rho_0)+O(\log q)}$ with $g(\rho_0) < 0$, and $g(\rho_0) \approx -\frac{2\sqrt{2}}{3}(1-\rho_0)^{3/2}$ near the threshold; Lemma 17 says that $h(0) \leq q^2\sqrt{\eta}$. So

$$2 \sin^2 \frac{\varphi}{4} \leq h(0) \lesssim q^2 e^{\frac{q}{2}g(\rho_0)},$$

and the right-hand side is small exactly when $1 - \rho_0 \gg ((\log \frac{1}{s} + \log q)/q)^{2/3}$, which is the choice made in (28). The exponent $2/3$ is the $3/2$ of g inverted, and the logarithm comes from the need to beat the polynomial factor q^2 of the smoothing estimate.

Remark 20 (What limits this rung). The proof uses the sequence only through Proposition 7(i)–(iv) and the moment conditions (22): it never uses that the segments are ordered in time, nor that the θ_j are positive, nor the lattice structure of the frequencies μ_m . This is why 4 is the ceiling of the method and not of the problem, and why the threshold $\tau = q$ (Section 7) is where the method stops.

6 The target angle

The proof is complete for $0 < \varphi \leq \pi$. This section disposes of the remaining angles, and identifies the one that matters in practice.

6.1 The reflection symmetry, and the two endpoints

The remaining angles are not a separate case: they are the mirror image of the ones treated, and the mirror is exact. The proof rests on two elementary facts about the free rotations, which are exact and cost nothing. First,

$$X(\pi)^{-1}Z(\theta)X(\pi) = Z(-\theta), \quad (34)$$

because conjugation by $X(\pi)$ sends σ_z to $-\sigma_z$; thus a Z evolution of *negative* angle is available at the cost of the corresponding positive angle, the two free π rotations that implement (34) being instantaneous. Consequently the model is unchanged if the angles θ_j are allowed to be any real numbers with the cost taken to be $\sum_j |\theta_j|$. Second, $X(2\pi) = -I$: a global sign is available for free as well.

Proof of Proposition 2. Let (θ, α) be admissible of order N for φ with cost T , and let U_λ be its interpolation, so that $U_1 = Z(\varphi)$ and $U_1^{(k)} = 0$ for $1 \leq k \leq N$. Consider $W_\lambda := X(2\pi)U_\lambda^{-1}$. Inverting the product (5) reverses the order of the factors and inverts each of them, and $Z(\lambda\theta_j)^{-1} = Z(-\lambda\theta_j)$ while $X(\alpha_j)^{-1} = X(-\alpha_j)$; hence W_λ is again of the form (5), with the angles $-\theta_L, \dots, -\theta_1$ in the reversed order, the X angles $-\alpha_j$, and one further free rotation $X(2\pi)$ at the left. By (34) each negative angle $-\theta_j$ is realised at cost θ_j , so W_λ is admissible in the original model with cost T . Moreover $W_1 = X(2\pi)Z(\varphi)^{-1} = (-I)Z(-\varphi) = Z(2\pi - \varphi)$, since $Z(2\pi - \varphi) = -Z(-\varphi)$, and the flatness is inherited: as $U_\lambda = U_1 + O((\lambda - 1)^{N+1})$ with U_1 unitary, also $U_\lambda^{-1} = U_1^{-1} + O((\lambda - 1)^{N+1})$, so $W_1^{(k)} = 0$ for $1 \leq k \leq N$. Thus every sequence admissible of order N for φ produces one admissible of order N for $2\pi - \varphi$ with the same cost, that is $T_{\min}(N, 2\pi - \varphi) \leq T_{\min}(N, \varphi)$; applying the same construction with φ replaced by $2\pi - \varphi$ gives the reverse inequality. $\square$

Proof of Corollary 3. If $0 < \varphi \leq \pi$ this is Theorem 1. If $\pi < \varphi < 2\pi$, put $\varphi' := 2\pi - \varphi \in (0, \pi)$; then Theorem 1 applied to φ' and Proposition 2 give the stated bound, and $\sin \frac{\varphi'}{4} = \sin(\frac{\pi}{2} - \frac{\varphi}{4}) = \cos \frac{\varphi}{4}$. $\square$

Remark 21 (Why the interval is open). Since $2\pi - 0 = 2\pi$, Proposition 2 gives $T_{\min}(N, 2\pi) = T_{\min}(N, 0)$, and both are 0 by Remark 10: the two angles at which the target $\pm I$ is the identity gate are mirror images of one another, and the range of the problem is exactly the open interval $(0, 2\pi)$. Equivalently, at $\varphi = 2\pi$ the bound (8) is not merely unproved but false, the zero-cost sequence of Remark 10 being admissible.

6.2 The case $\varphi = \pi$: the most demanding target

The requirement on the Z evolution is most severe at $\varphi = \pi$, and this is the case that occurs in practice. By Proposition 8,

$$\min_{A \in \mathbb{R}} \|X(A) - Z(\varphi)\| = 2 \sin \frac{\varphi}{4}, \quad (35)$$

the distance from the target to the set of propagators that the free rotations alone can produce. This distance increases with φ on $(0, \pi]$ and attains its largest value, $\sqrt{2}$, at $\varphi = \pi$: among all targets, $Z(\pi) = -i\sigma_z$ is the one that the free X rotations can least imitate. Equivalently, the guaranteed endpoint separation $h(0) \geq 2 \sin^2(\varphi/4)$ of Corollary 9 is largest at $\varphi = \pi$, where it equals 1; and since $\cos \frac{\pi}{2} = 0$, the identity $h(0) = 1 - \cos \frac{\varphi}{2} \cos \frac{A}{2}$ collapses there to

$$h(0) = 1 \quad \text{for every sequence, whatever its total } X \text{ angle}, \quad (36)$$

so that the criterion on which the whole proof rests holds in the strongest possible form: no choice of the free rotations brings the two endpoints of the interpolation closer than $\sqrt{2}$.

The three bounds of this paper therefore take their most explicit form at $\varphi = \pi$:

$$T \geq 2[\sqrt{2}(N+1)!]^{1/(N+1)} \quad (c^*(\pi) \geq 2/e), \quad (37)$$

$$T \geq 2[(2N+2)!]^{1/(2N+2)} \quad (c^*(\pi) \geq 4/e), \quad (38)$$

$$T \geq 4(N+1) \left(1 - C \left(\frac{\log(\sqrt{2}(2N+2))}{2N+2} \right)^{2/3} \right) \quad (c^*(\pi) \geq 4), \quad (39)$$

obtained from (13), (14) and (8) by $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\sin^2 \frac{\pi}{4} = \frac{1}{2}$. For the upper bound, the equiangular family (42) gives $T = (2N+1)\pi$ at $\varphi = \pi$ [4, 5], and the numerical optimisation of Section 7.3 gives the upper bounds collected in Table 1: $T = 8.150$ for $N = 1$, 13.347 for $N = 2$, 18.550 for $N = 3$, and up to $N = 12$ a ratio T/N falling from 8.15 into the range 5.6–6.1. Along the branch $N \leq 8$ the increments are 5.20, 5.20, 5.24, 5.27, 5.19, 5.45, 5.26, that is $T_N \approx 5.26N + 2.8$: a coefficient close to 5.2, not $2\pi \approx 6.28$. The window at $\varphi = \pi$ is thus

$$4 \leq c^*(\pi) \leq 2\pi \approx 6.28, \quad (40)$$

the numerical constructions lying near the middle of it. In particular the complexity of the problem at this angle is settled up to a constant factor:

$$4(N+1)(1 - o(1)) \leq T_{\min}(N, \pi) \leq (2N+1)\pi, \quad (41)$$

so that $T_{\min}(N, \pi) = \Theta(N)$, with the linear coefficient between 4 and 2π and numerically between 5.2 and 5.7 for the orders computed. This is the angle realised in the Rydberg-array protocol of [6], whose composite implements a conditional $Z(\pi)$ (Section 7); it is also the standard target of composite-pulse theory, a robust π rotation.

7 How sharp are the bounds?

7.1 The threshold $\tau = q$: the ceiling of the method

Lemma 18 needs $\rho_0 < 1$, that is $\tau < q$. At $\rho_0 = 1$ the two leading terms $\sqrt{1-z^2}$ and $\text{arccosh}(1/z)$ that make up g coincide and cancel, leaving $g = O((1-z)^{3/2})$; no exponential decay of the approximation error is then available. This is the exact threshold of the argument, and it is instructive to compute there. Take $\tau = q$ and $\rho = 1 + \varepsilon$ with $\varepsilon > 0$ small. Then $\frac{\tau}{2}(\rho - \rho^{-1}) - q \ln \rho = q \left[\frac{\rho - \rho^{-1}}{2} - \ln \rho \right] = \frac{q\varepsilon^2}{2}(1 + O(\varepsilon))$, so (26) gives $\eta \leq \frac{2}{\varepsilon} e^{q\varepsilon^2/2}(1 + o(1))$, least at $\varepsilon = q^{-1/2}$, where $\eta = O(\sqrt{q})$. Then $c_0 q^2 \sqrt{\eta} = O(q^{9/4})$, which is far larger than $2 \sin^2 \frac{\varphi}{4}$ for large q : the contradiction disappears. Hence $\tau = q$, that is $T = 2q = 4N + 4$,

so $T/N \rightarrow 4$, is the largest coefficient that the present method can reach. The Taylor route of Sections 3–4 never forms g , because it never uses the band; that is why it stops at $4/e$.

Two consequences are worth recording. First, the convergence in (8) is slow, of order $((\log \frac{q}{s})/q)^{2/3}$, and this is a property of the method rather than of the proof: since $g(\rho_0) \sim -\frac{2\sqrt{2}}{3}(1 - \rho_0)^{3/2}$ near the threshold, a polynomial factor such as q^2 in Lemma 17 can be overcome only by taking $1 - \rho_0 \gg ((\log \frac{q}{s})/q)^{2/3}$; improving the rate requires removing the square root in (25). Second, no argument that uses robustness only through the vanishing moments (22) can raise the coefficient above 4; raising it requires using further structure of (4), such as the positivity of the θ_j or the lattice of frequencies $\mu_m = \frac{1}{2} \sum_j \varepsilon_j \theta_j$.

7.2 Known constructions: $c^* \leq 2\pi$

The lower bound of Theorem 1 leaves a factor between 4 and the best known constructions, and those constructions transfer to the present model at the same cost. In [4] a composite pulse is a sequence of L primitive pulses of a *common* amplitude θ_0 with freely chosen, exactly controlled phases, followed by the target pulse of amplitude φ ; the error model scales all rotation angles by the same factor $1 + \varepsilon$, exactly as in (2), and the phases are exact, exactly as our X rotations are free. Writing the phases as X rotations and the pulses as our Z evolutions, their sequences are admissible in the sense of Model 6 at the same cost.

Their primary algebraic choice is $\theta_0 = 2\pi$ (Ref. [4], §III.1: the small correction rotation can be generated by the small pure error terms of a 2π pulse, and therefore θ_0 is set to 2π), which gives $T = 4\pi N + \varphi$; the π -pulse realisation obtained by *toggling* (Ref. [4], §V) halves the sequence duration, and Ref. [4], §VI reports those sequences as having $L = 2N + 1$ pulses of area π , the target pulse included. Hence

$$T_{\text{LYC}} = 2N\pi + \varphi, \quad (42)$$

the case $\varphi = \pi$ being $T = (2N + 1)\pi$, and therefore $c^*(\varphi) \leq 2\pi$ for every $0 < \varphi < 2\pi$, their solutions being available over a continuum of target angles. The two variants are simply the two ways of spending the same flatness budget: with the primary choice $\theta_0 = 2\pi$ every correcting pulse carries twice the rotation of the toggled one, and the cost doubles. The companion paper [5] exhibits the mechanism in closed form: with L equiangular gates the achievable response is a polynomial of degree L whose maximally flat member has flatness order $L + 1$, and the design point is $\theta = \pi$. Comparing with our $q = 2N + 2$, one π of rotation is spent per order of flatness, which is precisely the content of (42).

At $N = 1$ the construction can be exhibited and checked directly: the sequence

$$\theta = (\pi, \pi, \pi), \quad \alpha = \left(\frac{2\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}\right) \quad (43)$$

has cost $T = 3\pi$, implements $Z(\pi)$ and satisfies $dU_\lambda/d\lambda|_{\lambda=1} = 0$; a direct computation gives $\|U_1 - Z(\pi)\| < 10^{-15}$ and $\|U'_1\| < 10^{-11}$. The same family is available for every order and every target angle. Since $\sin \frac{\pi}{4} = 1/\sqrt{2}$ and $\cos \frac{\pi}{2} = 0$, the target $\varphi = \pi$ of Section 6.2 is the one for which the endpoint bound (19) is largest and, in fact, independent of the total X angle; it is also the target realised in the Rydberg-array protocol of [6], where the imperfect Z evolution is the free evolution under the distance-dependent interaction, the free X rotations are Rydberg pulses, each segment of free evolution is designed to accumulate the phase $V\tau = \pi$, and the composite implements the conditional $Z(\pi)$ of the CZ gate, with the control angles obtained from the recursion of Ref. [5] (Eq. (10) there).

7.3 Numerical optimisation: the upper end is not tight

Equation (42) is an artefact of the equiangular restriction $\theta_j \equiv \theta_0$, which [4] imposes for algebraic solubility — their constraint equations reduce to polynomial equations only when the amplitudes are equal, and to a maximally flat polynomial only when $\theta_0 = \pi$ or 2π ; the common amplitude is never optimised. Leaving the amplitudes free in the same model, and minimising T subject to (6) numerically, gives substantially smaller costs. The solutions below were obtained with a truncated-Taylor propagator

Table 1: Numerical upper bounds for $T_{\min}(N, \varphi)$ (heuristic optimiser; every entry re-verified with an independent propagator, nominal mismatch below 10^{-9} , deviation scaling as ε^{N+1}). “ratio” is T over the equiangular value (42).

φ	N	$(42) = 2N\pi + \varphi$	numerical T	ratio	numerical T/N
π	1	9.425	8.150	0.865	8.15
π	2	15.708	13.347	0.850	6.67
π	3	21.991	18.550	0.844	6.18
π	4	28.274	23.788	0.841	5.95
π	5	34.558	29.059	0.841	5.81
π	6	40.841	34.251	0.839	5.71
π	7	47.124	39.703	0.843	5.67
π	8	53.407	44.960	0.842	5.62
π	9	59.690	52.266	0.876	5.81
π	10	65.973	56.938	0.863	5.69
π	11	72.257	66.907	0.926	6.08
π	12	78.540	67.630	0.861	5.64
$\pi/2$	1	7.854	5.917	0.753	5.92
$\pi/2$	2	14.137	10.639	0.753	5.32
$\pi/2$	3	20.420	16.289	0.798	5.43

and a warm-started continuation in the segment number and in the robustness order: a time-budget parametrisation makes the minimisation a feasibility problem at fixed T , which is then pushed down by an adaptive march. For $N \geq 9$ the order step has to be seeded from the equiangular family, whose phases are computed at $\theta_j \equiv \pi$, and the resulting point is compressed by the same march. Each solution was re-verified with an independent propagator based on `scipy.linalg.expm`: the nominal mismatch is below 10^{-9} and the deviation from the target scales as ε^{N+1} over $0.5 \geq \varepsilon \geq 0.12$, confirming the order of robustness.

For $\varphi = \pi$, the most demanding target (Section 6.2), the increments $T_N - T_{N-1}$ are 5.20, 5.20, 5.24, 5.27, 5.19, 5.45, 5.26 for $N \leq 8$ (with $T_0 = \varphi$), so on that branch $T_N \approx 5.26N + 2.8$, i.e. a coefficient near 5.2 rather than $2\pi \approx 6.28$. The entries $N = 9, \dots, 12$ are *looser* upper bounds: they come from a different seeding of the order step (the equiangular family, whose phases are found numerically), and their optimisation depth, and hence their increment pattern, is not directly comparable — note the ratio 0.926 at $N = 11$ against 0.861 at $N = 12$, where the same march had a much better starting basin. What the extended table does establish is that (i) every T/N listed is strictly below 2π , the largest being 6.08 at $N = 11$, and (ii) whenever the numerics are given enough segments the quality returns to the 5.6–5.7 level of the $N \leq 8$ branch (5.69 at $N = 10$, 5.64 at $N = 12$). In the language of (42), the $N = 1$ optimum $8.150 = 2\theta_0 + \pi$ is what the same two-corrector-plus-target topology gives with a common amplitude $\theta_0 = 2.504 = 0.80\pi$ instead of π : the gain is already visible in the amplitude that [4] fixes for convenience.

These are numerical upper bounds from a heuristic optimiser, not proofs. They do not fix an asymptotic coefficient, but over $N \leq 12$ they leave little room for one near 2π : the equiangular restriction is the direction in which the upper end must move, and the coefficient indicated by the numerics is 5.2–5.7.

8 Discussion

Theorem 1 and Corollary 3 leave the window (15), that is $4 \leq c^*(\varphi) \leq 2\pi$ for every $0 < \varphi < 2\pi$. The lower end is the exact ceiling of every argument that uses robustness only through the vanishing moments (22), so it cannot be raised by sharpening the estimates of Lemmas 17 and 18; it can only be raised by using further structure of (4). The upper end is not tight, as the numerical table shows. Closing the

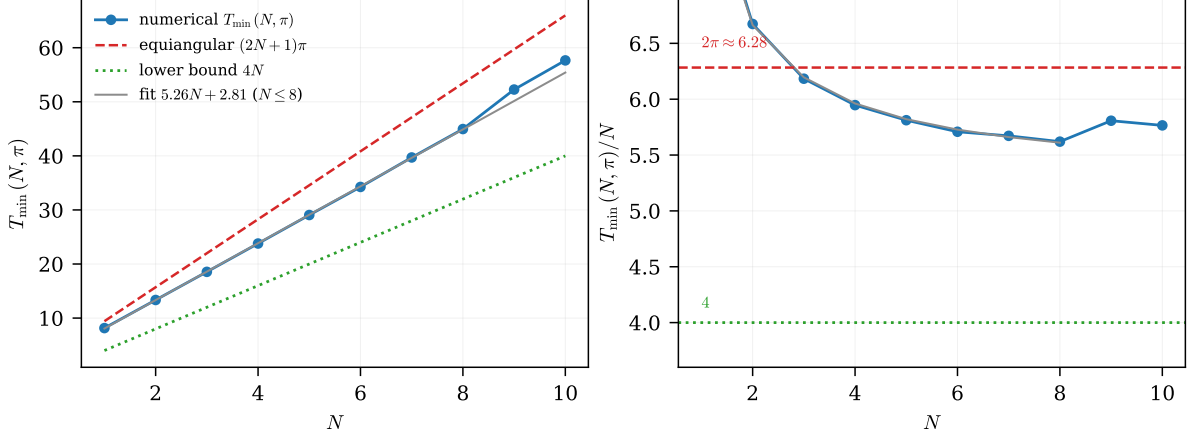


Figure 1: Numerical upper bounds for $\varphi = \pi$. Left: $T_{\min}(N, \pi)$ against the equiangular value $(2N + 1)\pi$ and the lower bound $4N$; the straight line is the fit $T \approx 5.26N + 2.8$ to the branch $N \leq 8$, where the increments are 5.20, 5.20, 5.24, 5.27, 5.19, 5.45, 5.26. Right: the same data divided by N , between the lower bound 4 and the equiangular coefficient 2π .

window therefore requires either an explicit construction better than the equiangular family (42), or a lower bound exploiting that extra structure.

Two further questions are natural. The first is the rate: the bound (8) approaches its linear asymptote only like $(\log N/N)^{2/3}$, and improving the rate requires a smoothing kernel with a better L^1 bound than (25); the square root there comes from the interpolation between the L^1 and L^∞ bounds on $\hat{\kappa}$, and removing it is a question about the approximation of $e^{-i\mu}$ on a band by a Fourier transform of an integrable function rather than about polynomials. The second is the φ -dependence: Theorem 1 shows that the leading coefficient is 4 for every angle in the open interval, while the numerically optimal costs at $N = 3$ are $T/N \approx 5.43$ at $\varphi = \pi/2$, and $T/N \approx 5.6$ – 5.8 at $\varphi = \pi$ for $3 \leq N \leq 12$ (Table 1 and Figure 1). The lower bound (8) does not distinguish the two at leading order, so this difference is a property of the constructions rather than of the bound.

9 Conclusion

For the synthesis of $Z(\varphi)$ from an imperfect, costly Z evolution and exact, free X rotations, N th-order robustness forces the total Z-evolution time to satisfy (8) for every target angle $0 < \varphi \leq \pi$, hence $c^*(\varphi) \geq 4$ throughout that interval; the reflection $\varphi \mapsto 2\pi - \varphi$ is an exact symmetry of the model, so the same holds for the remaining angles, and the coefficient is the same for all of them because the angle enters the argument only through the constant $h(0) \geq 2 \sin^2(\varphi/4)$, which is also the squared distance between the two endpoints of the interpolation. At the two mirror-related endpoints $\varphi = 0$ and $\varphi = 2\pi$ the target is $\pm I$, the identity gate up to a global phase, so that the free rotations alone implement it at zero cost and no positive lower bound can hold.

Flatness is approximation power. The error carrier is an oscillation of bandwidth $\tau = T/2$ whose first $q = 2N + 2$ moments vanish; vanishing moments let the constant term be replaced on the band by any polynomial of degree $< q$ times $e^{i\mu}$; and realising that replacement by a smoothing kernel costs exactly the best uniform polynomial approximation of $e^{-i\mu}$ there. Estimating that error by the decay of Chebyshev coefficients in a Bernstein ellipse makes it exponentially small whenever the bandwidth is below the flatness order. Taylor's theorem at the physical value of the strength sees a single point and cannot resolve the two leading terms of the ellipse rate; using the whole band cancels them and doubles the coefficient from $4/e$ to 4. A task requiring q orders of flatness therefore needs a bandwidth of at least q .

The main consequence is conceptual: fast, precise control can suppress the effect of an uncertain

interaction, and it can improve the constant factor substantially, but it cannot make arbitrarily high-order robustification asymptotically cheaper than linear in the robustness order. The open problem is consequently a constant-factor one: how close can the total Z-evolution time be pushed toward the rigorous lower coefficient 4?

A Proof of the smoothing estimate

This appendix proves Lemma 17. The only point of substance is that the cut-off keeps the L^1 norm of κ proportional to the approximation error, with a factor that is polynomial in q ; the analysis is otherwise bookkeeping.

Let P have degree $< q$ and put $\Psi(\mu) := 1 - P(\mu)e^{i\mu}$, so that $|\Psi| \leq \eta$ on $[-\tau, \tau]$, and assume $1 \leq \tau \leq q$, $q \geq 2$, $\eta \leq 1$ as in Lemma 17. Choose

$$\delta := \frac{\eta}{4q^2} \in \left(0, \frac{1}{16}\right], \quad \Omega := (1 + \delta)\tau \leq 2\tau, \quad (44)$$

and let $\chi \in C^\infty(\mathbb{R})$ satisfy $\chi = 1$ on $[-\tau, \tau]$, $\chi = 0$ outside $[-\Omega, \Omega]$, $0 \leq \chi \leq 1$, with

$$\int_{\mathbb{R}} |\chi'| \leq 4, \quad \int_{\mathbb{R}} |\chi''| \leq \frac{4}{\delta\tau}. \quad (45)$$

Such a χ exists for every $\delta > 0$, with a smooth monotone transition of width $\delta\tau$ at each end. Set

$$\hat{\kappa} := \Psi\chi, \quad \kappa(t) := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\kappa}(\mu) e^{-i\mu t} d\mu. \quad (46)$$

Since $\hat{\kappa}$ is smooth and compactly supported, κ is continuous and integrable with $\kappa(t) = O(t^{-2})$. On $[-\tau, \tau]$ we have $\chi \equiv 1$, so $\hat{\kappa} = \Psi = 1 - P(\mu)e^{i\mu}$; therefore (23), (24) and Lemma 16 give $h(0) = \sum_m c_m \hat{\kappa}(\mu_m) = \int_{\mathbb{R}} h\kappa$, and

$$|h(0)| \leq \|\kappa\|_{L^1} \|h\|_\infty \leq 2\|\kappa\|_{L^1}. \quad (47)$$

All norms and integrals in the rest of the proof are taken over the support $[-\Omega, \Omega]$ of $\hat{\kappa}$ unless stated otherwise.

Step 1: bounds on $\hat{\kappa}$. Let $\Theta := \sup_{[-\Omega, \Omega]} |\Psi'|$ and $\tilde{\eta} := \eta + \delta\tau\Theta$. By the mean value theorem, $|\Psi| \leq \tilde{\eta}$ on $[-\Omega, \Omega]$, because every point there is within $\delta\tau$ of the band $[-\tau, \tau]$. Hence

$$\sup |\hat{\kappa}| \leq \tilde{\eta}, \quad \int_{\mathbb{R}} |\hat{\kappa}| \leq 2\Omega \tilde{\eta} \leq 4\tau \tilde{\eta}. \quad (48)$$

For the second derivative, $\hat{\kappa}'' = \Psi''\chi + 2\Psi'\chi' + \Psi\chi''$. On the band $[-\tau, \tau]$ we have $\chi \equiv 1$ and only Ψ'' contributes; on the two transition regions, of total length $2\delta\tau$, the remaining contributions are bounded by $\sup_{[-\Omega, \Omega]} |\Psi''|$ against the length, by 2Θ against $\int |\chi'| \leq 4$, and by $\tilde{\eta}$ against $\int |\chi''| \leq 4/(\delta\tau)$. Consequently

$$\int_{\mathbb{R}} |\hat{\kappa}''| \leq A + 2\delta\tau B + 8\Theta + \frac{4\tilde{\eta}}{\delta\tau}, \quad A := \int_{[-\tau, \tau]} |\Psi''|, \quad B := \sup_{[-\Omega, \Omega]} |\Psi''|. \quad (49)$$

Since $\Psi = 1 - P(\mu)e^{i\mu}$,

$$\Psi' = -(P' + iP)e^{i\mu}, \quad \Psi'' = -(P'' + 2iP' - P)e^{i\mu}, \quad (50)$$

so $|\Psi'| \leq |P'| + |P|$ and $|\Psi''| \leq |P''| + 2|P'| + |P|$.

Step 2: bounds on P . On $[-\tau, \tau]$ we have $|P| = |e^{-i\mu} - \Psi| \leq 1 + \eta \leq 2$. Markov's inequality is applied on $[-\Omega, \Omega]$ itself, the interval on which Θ is defined, so that no extrapolation is needed:

$$\|P'\|_{\infty, [-\Omega, \Omega]} \leq \frac{q^2}{\Omega} \|P\|_{\infty, [-\Omega, \Omega]}, \quad \|P\|_{\infty, [-\Omega, \Omega]} \leq 2 + \delta\tau \|P'\|_{\infty, [-\Omega, \Omega]}, \quad (51)$$

the second because P is Lipschitz with constant $\|P'\|_{\infty, [-\Omega, \Omega]}$ over the extra band of width $\delta\tau$ at each end. Writing $m := \|P'\|_{\infty, [-\Omega, \Omega]}$, the two inequalities combine into $m \leq \frac{q^2}{\Omega}(2 + \delta\tau m)$, and $\frac{q^2\delta\tau}{\Omega} \leq q^2\delta = \frac{\eta}{4} \leq \frac{1}{4}$, so

$$m \leq \frac{2q^2}{\Omega} \left(1 - \frac{1}{4}\right)^{-1} \leq \frac{3q^2}{\tau}, \quad \|P\|_{\infty, [-\Omega, \Omega]} \leq 2 + \frac{3}{4}\eta \leq 3, \quad (52)$$

using $\Omega \geq \tau$. Markov's inequality applied once more, to P' of degree $< q$, gives $\|P''\|_{\infty, [-\Omega, \Omega]} \leq \frac{q^2}{\Omega}m \leq \frac{3q^4}{\tau^2}$, so

$$\Theta \leq 3 + \frac{3q^2}{\tau} \leq \frac{6q^2}{\tau}, \quad B \leq \frac{3q^4}{\tau^2} + \frac{6q^2}{\tau} + 3 \leq \frac{12q^4}{\tau^2}, \quad A \leq 2\tau B \leq \frac{24q^4}{\tau}, \quad (53)$$

where we used $\tau \geq 1$ and $\tau \leq q$ (which give $3 \leq 3q^2/\tau$ and $q^2 \leq q^4/\tau$). Moreover $\delta\tau\Theta \leq \frac{\eta}{4q^2} \cdot \tau \cdot \frac{6q^2}{\tau} = \frac{3\eta}{2}$, so

$$\tilde{\eta} = \eta + \delta\tau\Theta \leq \frac{5}{2}\eta, \quad \int_{\mathbb{R}} |\hat{\kappa}| \leq 2\Omega\tilde{\eta} \leq 4\tau \cdot \frac{5}{2}\eta = 10\tau\eta, \quad (54)$$

where we used $\Omega \leq 2\tau$ and (48). Substituting (53) into (49), and using $\eta \leq 1$, $\delta\tau \leq 1$, $\tau \leq q$ and $\frac{4\tilde{\eta}}{\delta\tau} = \frac{16q^2}{\eta\tau}\tilde{\eta} \leq \frac{40q^2}{\tau}$,

$$\int_{\mathbb{R}} |\hat{\kappa}''| \leq \frac{24q^4}{\tau} + \frac{6\eta q^2}{\tau} + \frac{48q^2}{\tau} + \frac{40q^2}{\tau} \leq c_1 \frac{q^4}{\tau}, \quad c_1 = 120, \quad (55)$$

where the second term is $2\delta\tau B \leq 2 \cdot \frac{\eta}{4q^2} \cdot \tau \cdot \frac{12q^4}{\tau^2} = \frac{6\eta q^2}{\tau}$, the third is $8\Theta \leq \frac{48q^2}{\tau}$, and the last three terms were bounded by $q^2 \leq q^4$ (as $q \geq 1$).

Step 3: L^1 bound on κ . Two integrations by parts, using that $\hat{\kappa}$ and $\hat{\kappa}'$ vanish outside $[-\Omega, \Omega]$, give $|\kappa(t)| \leq \frac{1}{2\pi} \int |\hat{\kappa}|$ and $|\kappa(t)| \leq \frac{1}{2\pi t^2} \int |\hat{\kappa}''|$. Hence for every $R > 0$

$$\|\kappa\|_{L^1} \leq \frac{R}{\pi} \int |\hat{\kappa}| + \frac{1}{\pi R} \int |\hat{\kappa}''| \leq \frac{2}{\pi} \left(\int |\hat{\kappa}| \right)^{1/2} \left(\int |\hat{\kappa}''| \right)^{1/2} \leq \frac{2\sqrt{10}c_1}{\pi} q^2 \sqrt{\eta}, \quad (56)$$

the second step by optimising over R and the third by (54) and $\int |\hat{\kappa}''| \leq c_1 q^4/\tau$. With (47) this is (25) with $c_0 = \frac{4}{\pi}\sqrt{10}c_1 = \frac{4}{\pi}\sqrt{1200} < 45$; taking $c_0 = 45$ completes the proof under the stated hypotheses $1 \leq \tau \leq q$, $q \geq 2$, $\eta \leq 1$.

Remark 22. Nothing in the proof except Step 2 uses $\tau \leq q$; for $1 \leq \tau \leq Cq^2$ the same chain gives $|h(0)| \leq c'_0 q^2 \sqrt{\eta}$ with a larger absolute c'_0 , because the bound $A \leq 2\tau B$ of (53) remains $O(q^4/\tau)$ as long as $\tau \leq Cq^2$. For $\tau \gg q^2$ the estimate $\int |\hat{\kappa}| \leq 4\tau\tilde{\eta}$ dominates and the conclusion fails; that regime is irrelevant here, since Theorem 1 only uses $1 \leq \tau \leq \rho_0 q < q$.

B Deferred computations

B.1 The derivative bound and the Taylor remainder

Proof of Lemma 12. For a single factor, $\frac{d^m}{d\lambda^m} Z(\lambda\theta_j) = \left(\frac{-i\theta_j\sigma_z}{2}\right)^m Z(\lambda\theta_j)$ has operator norm $(\theta_j/2)^m$, while the X factors are independent of λ and unitary. The product rule for the k -th derivative of a product of L one-parameter families therefore gives

$$U_\lambda^{(k)} = \sum_{k_1 + \dots + k_L = k} \frac{k!}{k_1! \dots k_L!} X(\alpha_L) Z^{(k_L)}(\lambda\theta_L) \dots X(\alpha_1) Z^{(k_1)}(\lambda\theta_1), \quad (57)$$

and taking norms term by term and summing the multinomial coefficients yields

$$\|U_\lambda^{(k)}\| \leq \sum_{k_1 + \dots + k_L = k} \frac{k!}{k_1! \dots k_L!} \prod_{j=1}^L (\theta_j/2)^{k_j} = \left(\sum_{j=1}^L \frac{\theta_j}{2} \right)^k = \tau^k.$$

Finally $h^{(k)} = -\frac{1}{2} \text{Tr}[U_1^\dagger U_\lambda^{(k)}]$ and $|\text{Tr } M| \leq 2\|M\|$ for a 2×2 matrix M — the trace is the sum of the two eigenvalues, each of modulus at most the largest singular value — so $|h^{(k)}| \leq \|U_\lambda^{(k)}\| \leq \tau^k$. $\square$

Proof of Lemma 13. Taylor's formula with the integral remainder, expanded about $\lambda = 1$ and evaluated at $\lambda = 0$, gives $U_0 = \sum_{k=0}^N \frac{(-1)^k}{k!} U_1^{(k)} + \frac{(-1)^{N+1}}{N!} \int_0^1 t^N U_t^{(N+1)} dt$; the terms $k = 1, \dots, N$ vanish by (6) and $U_1^{(0)} = U_1$. The estimate follows from Lemma 12 and $\int_0^1 t^N dt = 1/(N+1)$. $\square$

B.2 The ellipse estimate and the function g

Proof of Lemma 18. Rescale to $u = \mu/\tau \in [-1, 1]$ and let $f(u) := e^{-i\tau u}$. For $\rho > 1$ let $\mathcal{E}_\rho$ be the Bernstein ellipse with foci ± 1 and semi-axis sum ρ , and put $M(\rho) := \sup_{\mathcal{E}_\rho} |f|$. If $f = \sum_k a_k T_k$ is the Chebyshev expansion and f is bounded by $M(\rho)$ inside and on $\mathcal{E}_\rho$, then $|a_k| \leq 2M(\rho)\rho^{-k}$ for $k \geq 1$ [7, Ch. 8]; hence truncating at $k = q-1$ leaves an error of at most

$$\sum_{k \geq q} |a_k| \leq \frac{2M(\rho)\rho^{-q}}{1 - \rho^{-1}}. \quad (58)$$

The ellipse is parametrised by $u = \frac{1}{2}(\rho e^{i\theta} + \rho^{-1} e^{-i\theta})$, so $|\text{Im } u| \leq \frac{1}{2}(\rho - \rho^{-1})$ and $M(\rho) \leq \exp(\frac{\tau}{2}(\rho - \rho^{-1}))$. As $\sum_{k < q} a_k T_k(\mu/\tau)$ is a polynomial in μ of degree at most $q-1$, this gives (26).

For the second part, the critical point of $\frac{\tau}{2}(\rho - \rho^{-1}) - q \ln \rho$ solves $\tau(\rho^2 + 1) = 2q\rho$, that is $\rho = (q + \sqrt{q^2 - \tau^2})/\tau > 1$ exactly when $\tau < q$; at $\tau = \rho_0 q$ this reads $\rho = (1+w)/\rho_0$ with $w := \sqrt{1 - \rho_0^2}$. Since $\rho - \rho^{-1} = 2w/\rho_0$ and $\ln \frac{1+w}{\rho_0} = \text{arccosh}(1/\rho_0)$, the exponent at that ρ is

$$\frac{\tau}{2}(\rho - \rho^{-1}) - q \ln \rho = q \left(w - \ln \frac{1+w}{\rho_0} \right) = q g(\rho_0), \quad (59)$$

so the right-hand side of (26) becomes $(1 - \rho^{-1})^{-1} 2e^{qg(\rho_0)}$. The properties of g are Lemma 23 below. $\square$

Lemma 23. Let $g(z) = \sqrt{1 - z^2} - \text{arccosh}(1/z)$ for $z \in (0, 1)$. Then

- (i) $g(\cos \theta) = \sin \theta - \text{artanh}(\sin \theta)$ for $\theta \in (0, \frac{\pi}{2})$;
- (ii) $g(z) = -\sum_{k \geq 1} \frac{w^{2k+1}}{2k+1}$ with $w = \sqrt{1 - z^2}$, so $-\frac{w^3}{3(1-w^2)} \leq g(z) \leq -\frac{w^3}{3}$;
- (iii) $g(z) < 0$ and $g'(z) = \frac{\sqrt{1-z^2}}{z} > 0$ on $(0, 1)$, and $g(1) = 0$;
- (iv) $g(z) = -\frac{2\sqrt{2}}{3}(1-z)^{3/2} + O((1-z)^{5/2})$ as $z \uparrow 1$;
- (v) for $\rho = (1+w)/\rho_0$ with $0 < \rho_0 < 1$, $\frac{2}{1-\rho^{-1}} = \frac{2(1+w)}{1+w-\rho_0} \leq \frac{4}{w}$ and $w \geq \sqrt{1 - \rho_0^2}$;
- (vi) $g(\rho_0) \leq -\frac{1}{3}(1 - \rho_0)^{3/2}$.

Proof. (i) Put $z = \cos \theta$ with $\theta \in (0, \frac{\pi}{2})$, so $\sqrt{1 - z^2} = \sin \theta$. Then $\text{arccosh}(1/z) = \ln \frac{1+\sqrt{1-z^2}}{z} = \ln \frac{1+\sin \theta}{\cos \theta} = \text{artanh}(\sin \theta)$, the last identity because $\text{artanh } u = \frac{1}{2} \ln \frac{1+u}{1-u}$ and $\frac{1+\sin \theta}{1-\sin \theta} = \frac{(1+\sin \theta)^2}{\cos^2 \theta}$.

(ii) $\text{artanh } w = \sum_{k \geq 0} \frac{w^{2k+1}}{2k+1}$ for $|w| < 1$, so $g = w - \text{artanh } w = -\sum_{k \geq 1} \frac{w^{2k+1}}{2k+1}$. The upper bound is the $k = 1$ term, and for the lower bound $\sum_{k \geq 1} \frac{w^{2k+1}}{2k+1} \leq \frac{w^3}{3} \sum_{k \geq 0} w^{2k} = \frac{w^3}{3(1-w^2)}$.

(iii) Immediate from (ii) for the sign, since $w > 0$ for $z < 1$; and $g(1) = 0 - 0 = 0$. For the derivative, $\frac{d}{dz} \text{arccosh } \frac{1}{z} = \frac{1}{\sqrt{(1/z)^2 - 1}} \cdot (-\frac{1}{z^2}) = -\frac{1}{z\sqrt{1-z^2}}$, so $g'(z) = -\frac{z}{\sqrt{1-z^2}} + \frac{1}{z\sqrt{1-z^2}} = \frac{1-z^2}{z\sqrt{1-z^2}} = \frac{\sqrt{1-z^2}}{z} > 0$.

(iv) Write $z = 1 - \kappa$. Then $w = \sqrt{(1-z)(1+z)} = \sqrt{\kappa(2-\kappa)} = \sqrt{2\kappa}(1 + O(\kappa))$, and by (ii) $g = -\frac{w^3}{3}(1 + O(w^2)) = -\frac{(2\kappa)^{3/2}}{3}(1 + O(\kappa)) = -\frac{2\sqrt{2}}{3}\kappa^{3/2} + O(\kappa^{5/2})$.

- (v) $\frac{1}{\rho} = \frac{\rho_0}{1+w}$, so $1 - \frac{1}{\rho} = \frac{1+w-\rho_0}{1+w}$, and $1 + w - \rho_0 = (1 - \rho_0) + w \geq w$ while $1 + w \leq 2$. Finally $w = \sqrt{(1 - \rho_0)(1 + \rho_0)} \geq \sqrt{1 - \rho_0}$.
- (vi) By (ii), $g \leq -\frac{w^3}{3}$ with $w^2 = (1 - \rho_0)(1 + \rho_0) \geq (1 - \rho_0)$, so $w^3 \geq (1 - \rho_0)^{3/2}$. $\square$

B.3 Constants in the proof of Theorem 1

For completeness we record the numerical values produced by the two estimates of Section 5.2 and by the assembly. Lemma 17 gives $c_0 = 45$, and with $A = 14$ the contradiction argument gives $C = 14$ in (8) together with the absolute threshold $q_0 \approx 3.42 \times 10^7$, that is $N_0 \approx 1.71 \times 10^7$. The threshold is governed by the requirement that the exponent of q in (30) be negative, that is $A > 13$ for the constants produced here; the value $A = 14$ was chosen for convenience. None of these numbers is optimised, and the qualitative content of Theorem 1 is unaffected by them.

C Numerical solutions

Table 1 is reproduced below as explicit data: for every order $N \leq 12$ at $\varphi = \pi$ we list the Z -evolution times θ_j and the free X angles α_j , in units of π and in the order of (4). Sequences obtained with an extra left-most X pulse are printed with that pair of adjacent X rotations already merged, $X(a)X(b) = X(a+b)$, so that every printed sequence has exactly L Z 's and L X 's and is literally of the form (4). Each printed sequence was re-verified with the independent propagator of Section 7.3: the nominal mismatch is below 10^{-9} and the deviation from the target follows ε^{N+1} (the logarithmic slopes over $0.5 \geq \varepsilon \geq 0.12$ approach $N + 1$ from below).

$N = 1$ ($L = 17$, $T = 8.1505$, $T/N = 8.15$):

θ/π :

+0.1081, +0.0924, +0.0884, +0.0926, +0.1088, +0.3730, +0.1115, +0.0959, +0.0822, +0.0853
+0.1108, +0.3939, +0.1237, +0.1075, +0.1076, +0.1241, +0.3887

α/π :

+0.1395, +0.1459, +0.1459, +0.1393, +0.1235, -0.1265, -0.1450, -0.1505, -0.1227, -0.1440
-0.1276, +0.1510, +0.1695, +0.1747, +0.1693, +0.1495, -0.1229

$N = 2$ ($L = 21$, $T = 13.3466$, $T/N = 6.67$):

θ/π :

+0.1442, +0.8073, +0.1113, +0.0906, +0.0869, +0.0971, +0.4061, +0.1644, +0.1874, +0.5115
+0.2278, +0.1444, +0.1350, +0.0000, +0.3682, +0.0981, +0.0836, +0.0000, +0.0843, +0.1008
+0.3992

α/π :

+0.1016, +0.1662, +0.1948, +0.2051, +0.2019, +0.1815, -0.2755, -0.2851, -0.2415, -0.0902
-0.1457, -0.1611, -0.5999, +0.4443, +0.1900, +0.2169, +0.4475, -0.2234, +0.2152, +0.1912
-0.1145

$N = 3$ ($L = 25$, $T = 18.5505$, $T/N = 6.18$):

θ/π :

+0.1326, +0.1315, +0.1609, +0.2954, +0.1650, +0.0991, +0.0792, +0.0722, +0.0687, +0.0750
+0.2978, +0.0974, +0.0883, +0.0974, +0.1406, +0.5799, +0.2426, +1.0192, +0.3783, +0.0972
+0.0836, +0.0935, +0.3539, +0.1415, +0.9139

α/π :

-0.1286, -0.1233, -0.1027, -0.0819, -0.1239, -0.1464, -0.1591, +1.8381, -0.1426, -0.1385
+0.2096, +0.2321, +0.2318, +0.2106, +0.1565, +0.1998, +0.2150, +0.1166, +1.7383, -0.3016
-0.3042, -0.2702, +0.1827, +0.1750, +0.1215

$N = 4$ ($L = 31$, $T = 23.7878$, $T/N = 5.95$):

θ/π :

+0.4369, +0.4706, +0.1295, +0.1139, +0.1194, +0.1544, +0.1519, +0.1710, +0.1050, +0.0839
+0.0789, +0.0857, +0.0000, +0.2839, +0.0874, +0.0808, +0.0888, +0.1221, +0.6094, +0.2920
+0.5838, +0.2403, +0.2021, +0.8699, +0.3650, +0.0799, +0.0694, +0.0780, +0.3710, +0.5283
+0.5187

α/π :

+0.0824, -0.1321, -0.1503, -0.1541, -0.1495, -0.1057, -0.0978, -0.1595, -0.1865, +1.7988

-0.2007, -0.4610, +0.2822, +0.2075, +0.2328, +0.2382, +0.2097, +0.1511, +0.1823, +0.1680
+0.1650, +0.2089, +0.2021, +0.1837, +1.7281, -0.3104, -0.3117, -0.2761, +0.2396, -0.1346
+0.0372

$N = 5$ ($L = 38$, $T = 29.0593$, $T/N = 5.81$):

θ/π :

+0.3580, +0.1792, +0.1819, +0.2559, +0.2966, +0.1727, +0.1524, +0.1802, +0.3370, +0.1271
+0.0876, +0.0779, +0.0835, +0.2747, +0.0000, +0.0000, +0.0852, +0.0795, +0.0903, +0.1353
+0.2769, +0.1742, +0.1460, +0.1682, +0.0000, +0.6814, +0.1919, +0.1839, +0.8333, +0.5504
+0.7568, +0.0640, +0.0558, +0.0616, +0.3541, +0.8928, +0.1717, +0.5320

α/π :

-0.1125, -0.1256, -0.1127, -0.0924, -0.1397, -0.1595, -0.1602, -0.1339, -0.1530, -0.2064
-0.2305, -0.2330, +1.7902, -0.8157, -0.2869, -0.6726, +0.2498, +0.2462, +0.2175, +0.1533
+0.1503, +0.1898, +0.1899, +1.5084, +0.6510, +0.1041, +0.1201, +0.1116, +0.0741, -0.1809
+1.7263, -0.3102, -0.3129, -0.2816, +0.3706, +0.1407, +0.1381, -0.0246

$N = 6$ ($L = 48$, $T = 34.2512$, $T/N = 5.71$):

θ/π :

+0.7823, +0.5339, +0.1351, +0.1209, +0.1386, +0.1759, +0.2727, +0.2084, +0.1601, +0.1806
+0.3438, +0.1241, +0.0832, +0.0739, +0.0809, +0.2398, +0.0000, +0.0000, +0.0692, +0.0637
+0.0636, +0.0698, +0.0911, +0.1263, +0.1206, +0.1494, +0.1167, +0.1146, +0.1397, +0.0000
+0.8485, +0.5385, +0.2691, +0.1758, +0.1727, +0.5108, +0.8908, +0.6708, +0.0550, +0.0450
+0.0444, +0.0527, +0.2422, +0.1132, +0.7487, +0.1242, +0.1348, +0.4863

α/π :

+0.0909, -0.1320, -0.1346, -0.1527, -0.1438, -0.1170, -0.1201, -0.1628, -0.1724, -0.1516
-0.1552, -0.2163, -0.2398, -0.2419, +1.7801, -0.8361, -0.3070, -0.6922, +0.2061, +0.2100
+0.1936, +0.1601, +0.1151, +0.0762, +0.1030, +0.1718, +0.1750, +0.1651, +1.4948, +0.6371
+0.1405, +0.1001, +0.1542, +0.1679, +0.1563, -0.0792, -0.2384, +1.7795, -0.2567, -0.2687
-0.2600, -0.2267, +0.2208, +0.2101, +0.1103, +0.1130, +0.1020, -0.0460

$N = 7$ ($L = 55$, $T = 39.7030$, $T/N = 5.67$):

θ/π :

+0.0000, +0.3825, +0.2151, +0.2500, +0.5786, +0.2462, +0.2460, +0.3414, +0.2064, +0.1549
+0.1708, +0.2508, +0.1215, +0.0821, +0.0718, +0.0795, +0.2108, +0.0000, +0.0000, +0.0679
+0.0589, +0.0585, +0.0632, +0.0847, +0.1348, +0.1357, +0.1024, +0.1088, +0.1132, +0.1347
+0.2052, +0.0000, +0.3701, +0.0000, +0.2322, +0.2482, +0.5722, +0.2868, +1.0370, +0.1164
+0.9433, +0.7741, +0.3522, +0.1039, +0.4884, +0.1400, +0.1233, +0.0000, +0.0000, +0.0000
+0.0000, +0.3743, +0.4857, +0.5749, +0.5383

α/π :

+0.4938, -0.0991, -0.1237, -0.0917, -0.1629, -0.1838, -0.1569, -0.1653, -0.2149, -0.2149
-0.1868, -0.1913, -0.2299, -0.2506, -0.2533, +1.7865, -0.8353, -0.3082, -0.6937, +0.2099
+0.2147, +0.1975, +0.1639, +0.1087, +0.0517, +0.0937, +0.0836, +0.1893, +0.1827, +0.1651
+1.4783, +0.6186, +0.1306, +0.0333, +0.2105, +0.1682, +0.1308, +0.1516, +0.1222, +0.0176
+0.1190, +0.3914, +1.3865, -0.5667, -0.0800, -0.1322, -0.7192, -0.0043, -0.1842, +0.4683
+0.3474, +0.1368, -0.1018, +0.1523, +0.5309

$N = 8$ ($L = 58$, $T = 44.9604$, $T/N = 5.62$):

θ/π :

+0.0000, +0.0000, +0.1993, +0.2362, +0.5702, +0.1880, +0.0000, +0.1515, +0.1651, +0.2415
+0.2153, +0.1439, +0.1290, +0.1512, +0.2111, +0.1107, +0.0706, +0.0587, +0.0551, +0.0616
+0.2068, +0.0000, +0.0000, +0.0631, +0.0561, +0.0577, +0.0687, +0.1031, +0.1850, +0.1512
+0.1048, +0.1036, +0.1262, +0.2252, +0.3874, +0.0000, +0.2416, +0.3262, +0.4091, +0.3092
+0.6982, +0.2788, +1.0977, +0.9999, +0.7275, +0.3260, +0.0916, +0.4658, +0.1215, +0.0807
+0.0000, +0.0000, +0.0000, +0.3819, +0.4986, +1.5103, +0.1718, +0.7768

α/π :

+0.0694, +0.6253, -0.1324, -0.1221, -0.1397, +0.2677, -0.4405, -0.1779, -0.1619, -0.1259
-0.1578, -0.1802, -0.1826, -0.1465, -0.1474, -0.1877, -0.2098, -0.2238, -0.2025, +1.8100
-0.8371, -0.3069, -0.6775, +0.2084, +0.2122, +0.1988, +0.1712, +0.1164, +0.1178, +0.1797
+0.1887, +0.1933, +0.1669, +0.1046, +1.5286, +0.6546, +0.1825, +0.1317, +0.1728, +0.1617
+0.1207, +0.1438, +0.2295, +0.1035, +0.3893, +1.3851, -0.5672, -0.0813, -0.1348, -0.7316
-0.1805, -0.4700, +0.3483, +0.1291, -0.1354, +0.0785, +0.0607, +0.8060

$N = 9$ ($L = 63$, $T = 52.2657$, $T/N = 5.81$):

θ/π :

+0.0000, +0.0000, +0.0304, +0.3022, +0.3593, +0.5818, +0.2648, +0.0008, +0.1370, +0.1452
+0.2035, +0.1083, +0.1590, +0.1087, +0.1013, +0.1346, +0.1923, +0.0925, +0.0552, +0.0508
+0.0513, +0.0597, +0.1920, +0.0000, +0.0000, +0.0624, +0.0549, +0.0563, +0.0701, +0.1033

+0.1751, +0.1409, +0.0917, +0.0793, +0.1144, +0.2591, +0.3963, +0.0000, +0.2932, +0.5230
+0.4750, +0.4907, +0.5378, +0.6479, +0.6628, +1.1455, +0.8839, +0.6537, +0.2904, +0.0804
+0.4127, +0.1069, +0.0846, +0.0000, +0.0000, +0.0005, +0.3421, +0.1561, +0.4818, +2.1109
+0.1741, +0.4577, +0.6902

α/π :

+0.1762, +0.1523, +0.3015, -0.0565, -0.2891, -0.0945, +0.2492, -0.4572, -0.1883, -0.1799
-0.1667, -0.0587, -0.1574, -0.1582, -0.1567, -0.1302, -0.1494, -0.1897, -0.2095, -0.2203
-0.1933, +1.8395, -0.8335, -0.3037, -0.6717, +0.2143, +0.2185, +0.2051, +0.1741, +0.1114
+0.1005, +0.1874, +0.2005, +0.2021, +0.1575, +0.0650, +1.5701, +0.6492, +0.2728, +0.2617
+0.2898, +0.3958, -0.1752, -0.2972, +0.2261, +0.2285, +0.1079, +0.3884, +1.3839, -0.5681
-0.0822, -0.1351, -0.7310, -0.1800, +0.4704, +0.3487, +0.1308, -0.0693, -0.1526, +0.5584
-0.0194, +0.2010, +0.4299

$N = 10$ ($L = 70$, $T = 56.9380$, $T/N = 5.69$):

θ/π :

+0.1358, +0.3543, +0.0984, +0.0945, +0.3255, +0.1237, +0.3120, +0.0948, +0.3849, +0.1498
+0.1470, +0.2052, +0.4987, +0.3172, +0.1977, +0.1783, +0.2411, +0.3573, +0.3690, +0.0600
+0.1512, +0.1336, +0.1012, +0.3796, +0.7192, +0.2951, +0.5020, +0.5134, +0.2327, +0.2327
+0.4694, +0.2657, +0.2657, +0.4081, +0.1768, +0.1766, +0.1710, +0.1710, +0.3540, +0.2229
+0.2229, +0.1944, +0.1283, +0.2601, +0.1286, +0.1383, +0.3175, +0.1546, +0.1553, +0.5084
+0.1178, +0.1513, +0.2446, +0.2434, +0.2136, +0.1608, +0.1597, +0.1411, +0.1369, +0.1890
+0.1942, +0.1804, +0.1805, +0.6911, +0.5964, +0.5314, +0.2815, +0.4047, +0.3070, +0.2033

α/π :

+0.0871, -2.3822, -0.4261, -0.3905, +0.1951, -1.8188, -0.0455, -0.0635, +0.1969, +0.2118
+0.1933, +0.1485, +0.1669, +0.1523, +0.0666, +0.1360, -1.8310, -0.0122, +0.0998, +0.1494
+0.6600, +0.0904, +0.0469, +1.5504, -0.4001, -2.4494, -0.0304, +1.8833, -0.0046, -0.0042
+2.1149, -0.0001, +0.1278, +0.7046, +0.2152, -0.0267, -0.0010, -0.7323, -0.2973, -0.0184
-0.4691, -0.1894, -0.1334, +0.0299, +0.0379, +0.1087, -0.0394, -0.0220, -0.0436, +0.0379
-0.0286, +0.0540, +0.0136, -0.0941, -0.5751, -0.1015, +0.0109, +0.2153, +0.5154, +0.2779
+0.2165, +0.0381, -1.5422, -0.3238, +0.0341, -0.0731, -0.0314, +0.0551, +0.0354, +3.9414

$N = 11$ ($L = 75$, $T = 66.9068$, $T/N = 6.08$):

θ/π :

+0.1744, +0.1464, +0.1982, +0.1755, +0.2388, +0.1286, +0.1291, +0.2395, +0.2361, +0.2304
+0.2262, +0.2410, +0.2421, +0.2421, +0.2420, +0.2412, +0.2393, +0.2182, +0.2190, +0.2180
+0.2184, +0.2318, +0.2270, +0.4585, +0.2575, +0.2577, +0.2522, +0.1291, +0.1332, +0.2143
+0.2184, +0.3777, +0.4570, +0.2788, +0.2773, +0.4713, +0.4719, +0.2522, +0.2440, +0.4742
+0.2206, +0.2207, +0.4460, +0.4643, +0.4667, +0.8783, +0.4734, +0.2444, +0.1302, +0.1299
+0.9348, +0.2215, +0.2163, +0.2229, +0.2218, +0.2419, +0.2423, +0.1238, +0.1238, +0.2422
+0.2383, +0.2389, +0.1228, +0.1217, +0.2381, +0.4525, +0.2243, +0.2243, +0.4753, +0.2465
+0.2460, +0.4875, +0.4837, +0.4448, +0.4579

α/π :

+0.1846, +0.1104, -0.0754, -0.5387, -0.0476, -0.0036, -0.0259, -0.0745, +1.3112, -0.1231
-0.0823, -0.0103, -2.2654, -0.0055, -0.0104, -0.0229, -0.5519, -0.0247, +0.0570, +0.0096
+0.7686, +0.0193, +0.0729, +2.1638, -0.0058, -0.0345, -0.0874, -0.0274, -0.5983, -0.0494
+0.1204, -1.2137, +0.1149, -0.0129, +0.1086, +0.0071, -0.0195, +0.0177, +0.0191, -0.3737
-0.0038, +0.0284, +0.9608, +0.0248, +0.3007, +1.5426, -0.0359, -0.0104, +0.0074, -0.2106
+1.1467, -0.0555, -0.0760, +0.0225, +0.3595, +0.0086, -0.0027, +0.0004, +0.0046, +0.2144
+0.0102, -0.0128, -0.0192, -0.0131, -0.8614, -0.0259, +0.0046, +0.5237, +0.0147, +0.0016
+0.1511, +0.0299, -0.0403, -0.1559, -0.7436

$N = 12$ ($L = 78$, $T = 67.6302$, $T/N = 5.64$):

θ/π :

+0.8692, +0.2910, +0.2919, +0.2039, +0.3135, +0.4751, +0.4865, +0.4039, +0.3492, +0.4202
+0.2549, +0.2549, +0.1647, +0.1523, +0.3966, +0.2337, +0.2225, +0.2026, +0.1282, +0.1061
+0.0462, +0.1549, +0.1544, +0.5043, +0.2855, +0.2922, +0.2665, +0.2581, +0.1380, +0.1377
+0.2664, +0.4061, +0.3103, +0.2207, +0.2356, +0.2513, +0.2525, +0.3407, +0.3126, +0.3543
+0.5249, +0.2197, +0.2046, +0.4522, +0.4173, +0.6571, +0.1524, +0.1317, +0.1323, +0.4211
+0.2155, +0.4893, +0.2924, +0.1527, +0.1557, +0.4654, +0.1543, +0.1210, +0.3056, +0.0843
+0.0990, +0.1496, +0.3486, +0.2109, +0.2058, +0.4936, +0.1369, +0.1370, +0.6767, +0.1555
+0.1320, +0.3215, +0.1610, +0.1679, +0.3010, +0.3099, +0.1547, +0.2076

α/π :

+0.1806, +0.0057, -0.1432, -0.0917, -0.0607, -0.0193, +0.2837, -0.1281, -0.7897, -0.0690
+0.0050, -0.0860, -0.0791, -0.0835, +0.2380, -0.0150, +0.0074, +0.1837, -1.2634, +0.1764
+0.0863, -0.0074, -0.4093, -0.0025, -0.0688, -0.3214, -0.0540, -0.0273, +0.0087, +0.0201

-1.6335, -0.2273, +1.3088, -0.1371, -0.0401, -0.0086, -0.2964, +0.0388, -1.5772, +0.1500
-1.7774, +0.2213, +0.1500, +0.0459, +0.0770, +0.1364, +0.2307, +0.3011, +0.1869, -0.3178
-0.3239, -0.1548, -0.3085, -0.1849, -0.2542, -0.2897, -0.3971, -0.3149, +0.2601, +0.3061
-1.6330, +0.2379, +0.2108, +0.2357, +0.1940, -0.0447, -0.0208, -0.1105, +4.3610, +0.2697
+0.2907, -0.0754, +1.7694, -0.1580, -0.0510, -0.2320, -0.3778, -2.3293

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