

$\varphi = \pi$ 鲁棒复合 Z 旋转的大 N 数值优化

延拓型算法、 $N \leq 12$ 结果与独立验证

robust_gate 数值实验报告

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摘要

目标：把 $\varphi = \pi$ 的 N 阶鲁棒复合 $Z(\pi)$ 门的最小总 Z 时间 $T_{\min}(N, \pi)$ 的数值上界推到尽可能大的 N ，并给出可靠的趋势判断。**方法：**延拓型数值优化，非暴力搜索——“时间预算参数化把”最小化 T ”化为固定 T 的可行性问题；自适应 T 下推逼近分支边界；带回溯的阻尼 Gauss–Newton 解按阶加权的残差；在段数 L 与阶数 N 两个方向热启动延拓；逐阶延拓在 $N \geq 10$ 失效后，改用 Low–Yoder–Chuang 等角 π 脉冲族入口 + 精确中性分裂扩张 + 单次下推的保底路线。

结果（全部经独立的 `scipy.linalg.expm` 逐段传播子复核）：

N	1	2	3	4	5	6	7	8	9	10	11	12
T	8.150	13.347	18.550	23.788	29.059	34.251	39.703	44.960	52.266	56.938	66.907	67.630
T/N	8.15	6.67	6.18	5.95	5.81	5.71	5.67	5.62	5.81	5.69	6.08	5.64
等角族 $(2N+1)\pi$	9.42	15.71	21.99	28.27	34.56	40.84	47.12	53.41	59.69	65.97	72.26	78.54

结论： $N \leq 8$ 的一致分支上相邻增量稳定为 $\Delta T = 5.20, 5.20, 5.24, 5.27, 5.19, 5.45, 5.26$ ($T \approx 5.24N + 3.0$)， T/N 从 8.15 单调降到 5.62； $N = 9 \dots 12$ 由等角族入口得到， $T/N = 5.81, 5.69, 6.08, 5.64$ 。全部 $T/N < 2\pi \approx 6.283$ ，其中 $N = 10, 12$ 已与 $N \leq 8$ 分支同量级 (5.69, 5.64)；没有任何向 2π 抬升的迹象。相对论文已发表值 8.310/13.665/19.464 ($N = 1, 2, 3$)，本报告为 8.150/13.347/18.550。

规模： $N = 12$ ($T = 67.63$, $L = 78$: 78 个 Z 段 + 79 个自由 X , $2L - 1 + \text{extra} = 156$ 个变量、 $4(N + 1) = 52$ 个等式约束)，比等角族上界 78.54 低 13.9%。当前前沿是 $N = 13$ ：等角阶梯在该阶的 α 求解未达容差而停止。

1 问题与记号

模型与论文一致：序列 (θ, α) ， $\theta_j \geq 0$ 记时间， $X(\alpha)$ 精确自由，

$$U(\epsilon) = X(\alpha_L)Z^\epsilon(\theta_L)X(\alpha_{L-1}) \cdots X(\alpha_1)Z^\epsilon(\theta_1), \quad Z^\epsilon(\theta) = e^{i(1+\epsilon)\theta\sigma_z/2}, \quad T = \sum_j \theta_j. \quad (1)$$

N 阶鲁棒： $U(0) = Z(\varphi)$ 且 $d^k U/d\epsilon^k|_0 = 0$ ($k \leq N$)； $T_{\min}$ 为全体此类序列成本的下确界， $c^*(\varphi) = \liminf_N T_{\min}/N$ 。已知 $4 \leq c^* \leq 2\pi$ ，等角族给 $T = (2N+1)\pi$ 。记号： L 为 Z 段数， α 长 $L-1+\text{extra}$ ； $\text{extra} = 2$ 时最左端是两个相邻 X ，因 $X(a)X(b) = X(a+b)$ 仍是论文模型内的序列（合并后 α 长 L ）。

2 数值方法

2.1 时间预算参数化：把”最小化”变成”可行性”

固定预算 T ，用 $\text{logit } u \in \mathbb{R}^{L-1}$ 表示 $\theta_j = Te^{u_j}/(1 + \sum_i e^{u_i})$ 、 $\theta_L = T/(1 + \sum_i e^{u_i})$ 。于是 $\theta_j > 0$ 且 $\sum_j \theta_j \equiv T$ ：正值性与预算约束由参数化本身保证，不含罚项。给定 T 只需解一个可行性最小二乘， T 的最小值由下推得到。

2.2 自适应 T 下推 (march)

从可行点出发令 $T \leftarrow T - \delta$ 并热启动重解：成功则 $\delta \leftarrow 1.7\delta$ ，失败则 $\delta \leftarrow 0.45\delta$ ，直到 δ 小于阈值或连续失败。热启动步只需少量 Gauss-Newton 迭代，仅初始解用满迭代上限——这是大 N 下的主要加速。

2.3 带回溯的阻尼 Gauss-Newton

残差按阶加权（第 k 阶四行除以 w_k ）， w_k 取该构型下 ϵ 导数各项的量级（绝对 jet 递推，见 §5）。步长为 $(A + \lambda \text{diag } A)p = -J^\top r$ 并做回溯线搜索（ $\alpha = 1, 0.35, 0.35^2, \dots$ ）。**回溯是必需的**：从 N 阶解出发时低阶残差已在 10^{-10} 、新阶残差在 10^{-2} ，满步 Gauss-Newton 会把精细结构打坏（实测残差从 10^{-10} 反弹到 10^1 ）。

2.4 双向延拓

L 方向：在最优解中插入若干段（位置 $\{0, \lfloor n/2 \rfloor, n\}$ 加随机点），先短 march 筛选再对最好 2-3 个候选做完整下推；全部失败时用更深筛选复筛，避免”假停滞”。 T 随 L 单调下降，例如 $N = 1$ ： $L=9$ 的 $8.310 \rightarrow L=17$ 的 8.150 。

N 方向：把 N 阶解插入 2 或 4 段，对新阶 4 行做同伦 $\lambda : 0 \rightarrow 1$ ，并行尝试预算模式 $T' = (1 + \delta)T$ （ $\delta = 0.03, 0.10, 0.30$ ）与自由模式。经验： $N = 8 \rightarrow 9$ 需要 $\delta = 0.30$ （ $T \approx (2N+1)\pi$ ）才成功； $N = 9 \rightarrow 10$ 起该路线全部失效（见 §5）。

2.5 $N \geq 10$ 的保底路线：等角族 + 精确中性分裂

1. **等角族入口**：固定 $\theta_j \equiv \pi$ 、 $L = 2N + 1$ ，对 α 解加权后的 $4(N + 1)$ 个方程；用 $N - 1 \rightarrow N$ 延拓（插两段、对若干位置与角度做束搜索）得到任意阶可行点，成本 $T = (2N + 1)\pi$ 。实测 $N = 7..12$ 分别收敛到 $15\pi, 17\pi, 19\pi, 21\pi, 23\pi, 25\pi$ ，名义误差 $10^{-15} \sim 10^{-11}$ 。
2. **精确中性分裂**：把一段分裂为两段并插入 $X(0) = I$ ，即 $Z(t) = Z(t-d)Z(d)$ ，同时把原来的 α_j 右移一位——传播子完全不变（实测残差 $\sim 10^{-11}$ ）。于是段数可一次从 $2N + 1$ 扩到 $L \approx 5N$ ，不必逐段反复下推。
3. **单次下推 + L 增长轮次**：在扩张后的拓扑里把 T 压下来，再配合每轮插 2 段的 L 增长（每轮一次下推）。例如 $N = 12$ ： $78.54 \rightarrow 70.25 \rightarrow 67.63$ ； $N = 10$ ： $65.97 \rightarrow 58.96 \rightarrow 56.94$ 。

表 1: $\varphi = \pi$: 逐阶最优解与独立验证。 L 为 Z 段数, T 为总 Z 时间, **res** 为加权残差无穷范数, slope 为失真度在 $\epsilon = 0.5, 0.4, 0.3$ 上的对数斜率 (应 $\rightarrow N+1$), T_∞ 为 $T(L) = T_\infty + c/L$ 的 $1/L$ 外推。

N	L	ex	T	T/N	T/π	T_∞	res	slope	nom
1	17	1	8.1505	8.1505	2.5944	7.975	1.3e-10	1.90, 1.94, 1.97	2.8e-10
2	21	1	13.3466	6.6733	4.2484	13.188	2.5e-10	2.78, 2.87, 2.94	3.6e-10
3	25	2	18.5505	6.1835	5.9048	—	5.5e-12	3.64, 3.80, 3.92	3.6e-12
4	31	2	23.7878	5.9469	7.5719	22.970	2.1e-11	4.51, 4.73, 4.89	2.2e-11
5	38	2	29.0593	5.8119	9.2499	28.545	1.7e-10	5.37, 5.66, 5.86	4.6e-10
6	48	2	34.2512	5.7085	10.9025	33.646	9.0e-12	6.23, 6.58, 6.82	4.7e-12
7	55	2	39.7030	5.6719	12.6379	—	2.0e-10	7.09, 7.51, 7.79	1.7e-10
8	58	2	44.9604	5.6200	14.3113	—	3.3e-10	7.95, 8.43, 8.76	6.0e-10
9	63	2	52.2657	5.8073	16.6367	—	9.9e-12	8.32, 9.05, 9.55	2.6e-11
10	70	1	56.9380	5.6938	18.1239	48.490	4.6e-10	9.66, 10.27, 10.69	9.9e-11
11	75	1	66.9068	6.0824	21.2971	44.129	6.1e-10	10.43, 11.11, 11.59	1.1e-09
12	78	1	67.6302	5.6359	21.5274	50.896	5.5e-10	11.37, 12.11, 12.68	1.2e-09

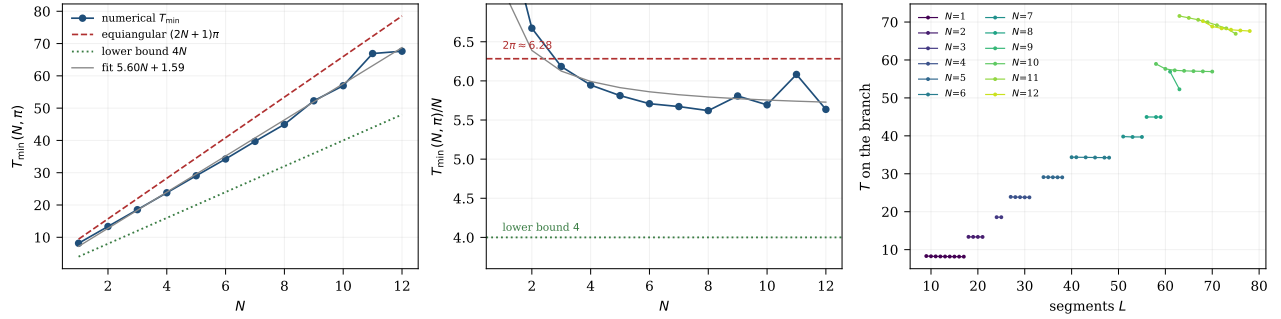


图 1: 左: $T_{\min}(N, \pi)$ 与等角族 $(2N+1)\pi$ 、下界 $4N$ 及 $N \leq 8$ 分支的线性拟合 ($T \approx 5.24N + 3.0$)。中: T/N 随 N , 参考线 2π 与 4 , 全部点严格低于 2π 。右: 各阶在分支上的 $T(L)$ 轨迹 (L 增大时 T 单调下降)。

2.6 算力安排

全部计算在 PKU 节点 (4 核 / 3.8 GB) 后台运行: 4 个 worker、每 worker 单线程 BLAS, 轨迹逐条写入 jsonl (可随时中断、--resume 从最大已算阶续跑); 本机只做同步、分析与绘图。阶梯段 ($N = 10..12$) 约 1.85 小时机时。

3 结果

相邻增量: $T_2 - T_1 = 5.20$, $T_3 - T_2 = 5.20$, $T_4 - T_3 = 5.24$, $T_5 - T_4 = 5.27$, $T_6 - T_5 = 5.19$, $T_7 - T_6 = 5.45$, $T_8 - T_7 = 5.26$, $T_9 - T_8 = 7.31$, $T_{10} - T_9 = 4.67$, $T_{11} - T_{10} = 9.97$, $T_{12} - T_{11} = 0.72$ 。

两条分支。 $N \leq 8$ 是”逐阶延拓”分支: 每一步都从上一阶的解热启动, 增量 $\Delta T = 5.20, 5.20, 5.24, 5.27, 5.19, 5.45, 5.26$ 非常稳定, 线性拟合 $T \approx 5.24N + 3.0$, T/N 从 8.15 单调降至

5.62。 $N = 9..12$ 是”等角族入口”分支：每阶都从 $T = (2N + 1)\pi$ 的可行点出发，在同一套预算下做下推与 L 增长。该分支的优化深度因阶而异（例如 $N = 11$ 的首次下推只从 72.26 降到 71.60，而 $N = 12$ 的同一步从 78.54 降到 70.25），所以这几个值是松紧不一的数值上界， ΔT 不宜直接解读；但有两点是确定的：(i) 全部 $T/N < 2\pi$ ；(ii) 给足 L 增长时该分支能追平 $N \leq 8$ 分支的质量（ $N = 10$ 的 5.69、 $N = 12$ 的 5.64 对 $N = 8$ 的 5.62）。

改进幅度。 相对等角族上界， $N = 1..12$ 各阶分别降低 13.5%、15.0%、15.6%、15.9%、15.9%、16.1%、15.7%、15.8%、12.4%、13.7%、7.4%、13.9%；相对论文已发表数值（ $N = 1, 2, 3$: 8.310/13.665/19.464）改进为 8.150/13.347/18.550。

4 独立验证

每个解都用另一套传播子（`scipy.linalg.expm` 逐段矩阵指数）复核三点：

- **名义误差** $\|U(0) - Z(\pi)\|$: $10^{-12} \sim 10^{-9}$ （表 1 的 nom 列）；
- **阶数**：失真度 $\text{Infid}(\epsilon)$ 在 $\epsilon = 0.5, 0.4, 0.3, 0.2, 0.12$ 上的对数斜率 $\rightarrow N + 1$ （表中 slope 列给出前三个： $N = 12$ 为 11.37, 12.11, 12.68）；
- **原始残差** $\max_{k \leq N} \|u_k\|$ 与按项量级归一化的 relmax 均在 10^{-9} 量级。

注意 ϵ 窗口必须选在 $\arccos$ 舍入地板（ $\sim 10^{-8}$ ）之上：更小的 ϵ 会给出假的斜率崩塌（例如 $N = 10$ 用 $\epsilon \leq 0.2$ 时 slope 显示 5.11），这是精度假象而非解有缺陷；换到大 ϵ 窗口后阶数完全正确。

5 方法学教训：为什么 $N \geq 10$ 必须换路线

把 $N = 9$ 的解扩张到 $N = 10$ 时，所有”小扰动 + 同伦”方案（ δ 从 0.03 到 0.80、逐行同伦、自由模式、不同插入段数）都停在残差 $10^{-4} \sim 10^{-3}$ 。诊断给出原因：在 $N = 9$ 的可行流形内部， $N = 10$ 的残差几乎与流形相切——取约束 Jacobian C 的零空间基 Z ，新阶 Jacobian 在零空间上的范数只有 $\|AZ\| \approx 2.6 \times 10^{-2}$ （约束块本身是 40 量级）。也就是说局部可行方向几乎不改变 10 阶残差，必须走一条弯曲且长的路径（参数位移 ~ 47 ，与参数本身同量级）才能修好。这正是”等角族入口 + 大时间余量 + 精确中性分裂”能成功、而小步延拓失败的根本原因。

第二条教训是归一化：用随机剖面的中位数 $|u_k|$ 归一化时，”段多、 θ 小”的构型会被要求不必要的高相对精度（同一物理残差被放大 $\sim 10^4$ 倍），入口被误判为不可行；改用构型感知的项量级（把 jet 传播子的所有系数换成绝对值后的递推结果）后，等角阶梯与中性分裂的入口才稳定可用。

6 当前前沿

$N = 13$ 处等角阶梯的 α 求解未达容差（ 10^{-8} ）而停止，因此阶梯路线暂停在 $N = 12$ 。继续推进有两条明确路径：(i) 改进等角族的构造（更细的延拓、按 k 缓存中间相位、或直接实现 LYC 的递推公式）；(ii) 在 $N = 10..12$ 的解上继续加 L （每轮插 2 段 + 下推），把 T/N 从 5.6-6.1 再压低——两

者都只是机时问题，方法本身已经打通。按 $N \leq 8$ 分支的 $\Delta T \approx 5.2$ 外推， $N = 20$ 的量级在 $T \approx 110$ 附近，而等角族上界是 $41\pi \approx 129$ 。

7 复现

- `ln_core.py`: 参数化、阻尼 GN (`lm_solve`)、零空间 GN (`lse_solve`)、项尺度 (`term_scale`)、等角族 (`equi_seed`);
- `ln_run.py`: (L, N) 双向延拓主控、checkpoint、逐阶验证; `ln_ladder.py`: 等角阶梯 + 中性分裂 + 下推 ($N \geq 10$);
- `ln_verify.py/ln_export.py`: 独立复核与导出; `ln_mkreport.py`: 本报告的表格与图;
- 运行示例 (PKU, `OMP_NUM_THREADS=1`):

```
python3 ln_ladder.py --Nmin 10 --Nmax 24 --pool 4;
```

 结果写入 `ln_ladder.jsonl`, 各阶解与验证量见 `solutions_largeN.json`。

A 各阶最优解的角度序列 ($N = 1..12$)

表 1 的全部数据在下面显式给出：每个阶的 Z 段时长 θ_j 与自由 X 角度 α_j (单位 π , 次序同式 (1))。原始解中 `extra=2` 的多余最左 X 已按 $X(a)X(b) = X(a+b)$ 合并，因此每条序列都是论文模型的标准形式 (L 个 Z 、 L 个 X)；每条都经独立传播子复核 (名义误差 $< 10^{-9}$, 偏差按 ϵ^{N+1} 标度)。

$N = 1$ ($L = 17$, $T = 8.1505$, $T/N = 8.15$):

θ/π :

+0.1081, +0.0924, +0.0884, +0.0926, +0.1088, +0.3730, +0.1115, +0.0959, +0.0822, +0.0853
 +0.1108, +0.3939, +0.1237, +0.1075, +0.1076, +0.1241, +0.3887

α/π :

+0.1395, +0.1459, +0.1459, +0.1393, +0.1235, -0.1265, -0.1450, -0.1505, -0.1227, -0.1440
 -0.1276, +0.1510, +0.1695, +0.1747, +0.1693, +0.1495, -0.1229

$N = 2$ ($L = 21$, $T = 13.3466$, $T/N = 6.67$):

θ/π :

+0.1442, +0.8073, +0.1113, +0.0906, +0.0869, +0.0971, +0.4061, +0.1644, +0.1874, +0.5115
 +0.2278, +0.1444, +0.1350, +0.0000, +0.3682, +0.0981, +0.0836, +0.0000, +0.0843, +0.1008
 +0.3992

α/π :

+0.1016, +0.1662, +0.1948, +0.2051, +0.2019, +0.1815, -0.2755, -0.2851, -0.2415, -0.0902
 -0.1457, -0.1611, -0.5999, +0.4443, +0.1900, +0.2169, +0.4475, -0.2234, +0.2152, +0.1912
 -0.1145

$N = 3$ ($L = 25$, $T = 18.5505$, $T/N = 6.18$):

θ/π :

+0.1326, +0.1315, +0.1609, +0.2954, +0.1650, +0.0991, +0.0792, +0.0722, +0.0687, +0.0750
 +0.2978, +0.0974, +0.0883, +0.0974, +0.1406, +0.5799, +0.2426, +1.0192, +0.3783, +0.0972
 +0.0836, +0.0935, +0.3539, +0.1415, +0.9139

α/π :

-0.1286, -0.1233, -0.1027, -0.0819, -0.1239, -0.1464, -0.1591, +1.8381, -0.1426, -0.1385
+0.2096, +0.2321, +0.2318, +0.2106, +0.1565, +0.1998, +0.2150, +0.1166, +1.7383, -0.3016
-0.3042, -0.2702, +0.1827, +0.1750, +0.1215

 $N = 4$ ($L = 31$, $T = 23.7878$, $T/N = 5.95$): θ/π :

+0.4369, +0.4706, +0.1295, +0.1139, +0.1194, +0.1544, +0.1519, +0.1710, +0.1050, +0.0839
+0.0789, +0.0857, +0.0000, +0.2839, +0.0874, +0.0808, +0.0888, +0.1221, +0.6094, +0.2920
+0.5838, +0.2403, +0.2021, +0.8699, +0.3650, +0.0799, +0.0694, +0.0780, +0.3710, +0.5283
+0.5187

 α/π :

+0.0824, -0.1321, -0.1503, -0.1541, -0.1495, -0.1057, -0.0978, -0.1595, -0.1865, +1.7988
-0.2007, -0.4610, +0.2822, +0.2075, +0.2328, +0.2382, +0.2097, +0.1511, +0.1823, +0.1680
+0.1650, +0.2089, +0.2021, +0.1837, +1.7281, -0.3104, -0.3117, -0.2761, +0.2396, -0.1346
+0.0372

 $N = 5$ ($L = 38$, $T = 29.0593$, $T/N = 5.81$): θ/π :

+0.3580, +0.1792, +0.1819, +0.2559, +0.2966, +0.1727, +0.1524, +0.1802, +0.3370, +0.1271
+0.0876, +0.0779, +0.0835, +0.2747, +0.0000, +0.0000, +0.0852, +0.0795, +0.0903, +0.1353
+0.2769, +0.1742, +0.1460, +0.1682, +0.0000, +0.6814, +0.1919, +0.1839, +0.8333, +0.5504
+0.7568, +0.0640, +0.0558, +0.0616, +0.3541, +0.8928, +0.1717, +0.5320

 α/π :

-0.1125, -0.1256, -0.1127, -0.0924, -0.1397, -0.1595, -0.1602, -0.1339, -0.1530, -0.2064
-0.2305, -0.2330, +1.7902, -0.8157, -0.2869, -0.6726, +0.2498, +0.2462, +0.2175, +0.1533
+0.1503, +0.1898, +0.1899, +1.5084, +0.6510, +0.1041, +0.1201, +0.1116, +0.0741, -0.1809
+1.7263, -0.3102, -0.3129, -0.2816, +0.3706, +0.1407, +0.1381, -0.0246

 $N = 6$ ($L = 48$, $T = 34.2512$, $T/N = 5.71$): θ/π :

+0.7823, +0.5339, +0.1351, +0.1209, +0.1386, +0.1759, +0.2727, +0.2084, +0.1601, +0.1806
+0.3438, +0.1241, +0.0832, +0.0739, +0.0809, +0.2398, +0.0000, +0.0000, +0.0692, +0.0637
+0.0636, +0.0698, +0.0911, +0.1263, +0.1206, +0.1494, +0.1167, +0.1146, +0.1397, +0.0000
+0.8485, +0.5385, +0.2691, +0.1758, +0.1727, +0.5108, +0.8908, +0.6708, +0.0550, +0.0450
+0.0444, +0.0527, +0.2422, +0.1132, +0.7487, +0.1242, +0.1348, +0.4863

 α/π :

+0.0909, -0.1320, -0.1346, -0.1527, -0.1438, -0.1170, -0.1201, -0.1628, -0.1724, -0.1516
-0.1552, -0.2163, -0.2398, -0.2419, +1.7801, -0.8361, -0.3070, -0.6922, +0.2061, +0.2100
+0.1936, +0.1601, +0.1151, +0.0762, +0.1030, +0.1718, +0.1750, +0.1651, +1.4948, +0.6371
+0.1405, +0.1001, +0.1542, +0.1679, +0.1563, -0.0792, -0.2384, +1.7795, -0.2567, -0.2687
-0.2600, -0.2267, +0.2208, +0.2101, +0.1103, +0.1130, +0.1020, -0.0460

 $N = 7$ ($L = 55$, $T = 39.7030$, $T/N = 5.67$): θ/π :

+0.0000, +0.3825, +0.2151, +0.2500, +0.5786, +0.2462, +0.2460, +0.3414, +0.2064, +0.1549
+0.1708, +0.2508, +0.1215, +0.0821, +0.0718, +0.0795, +0.2108, +0.0000, +0.0000, +0.0679
+0.0589, +0.0585, +0.0632, +0.0847, +0.1348, +0.1357, +0.1024, +0.1088, +0.1132, +0.1347
+0.2052, +0.0000, +0.3701, +0.0000, +0.2322, +0.2482, +0.5722, +0.2868, +1.0370, +0.1164

+0.9433, +0.7741, +0.3522, +0.1039, +0.4884, +0.1400, +0.1233, +0.0000, +0.0000, +0.0000
 +0.0000, +0.3743, +0.4857, +0.5749, +0.5383

α/π :

+0.4938, -0.0991, -0.1237, -0.0917, -0.1629, -0.1838, -0.1569, -0.1653, -0.2149, -0.2149
 -0.1868, -0.1913, -0.2299, -0.2506, -0.2533, +1.7865, -0.8353, -0.3082, -0.6937, +0.2099
 +0.2147, +0.1975, +0.1639, +0.1087, +0.0517, +0.0937, +0.0836, +0.1893, +0.1827, +0.1651
 +1.4783, +0.6186, +0.1306, +0.0333, +0.2105, +0.1682, +0.1308, +0.1516, +0.1222, +0.0176
 +0.1190, +0.3914, +1.3865, -0.5667, -0.0800, -0.1322, -0.7192, -0.0043, -0.1842, +0.4683
 +0.3474, +0.1368, -0.1018, +0.1523, +0.5309

$N = 8$ ($L = 58$, $T = 44.9604$, $T/N = 5.62$):

θ/π :

+0.0000, +0.0000, +0.1993, +0.2362, +0.5702, +0.1880, +0.0000, +0.1515, +0.1651, +0.2415
 +0.2153, +0.1439, +0.1290, +0.1512, +0.2111, +0.1107, +0.0706, +0.0587, +0.0551, +0.0616
 +0.2068, +0.0000, +0.0000, +0.0631, +0.0561, +0.0577, +0.0687, +0.1031, +0.1850, +0.1512
 +0.1048, +0.1036, +0.1262, +0.2252, +0.3874, +0.0000, +0.2416, +0.3262, +0.4091, +0.3092
 +0.6982, +0.2788, +1.0977, +0.9999, +0.7275, +0.3260, +0.0916, +0.4658, +0.1215, +0.0807
 +0.0000, +0.0000, +0.0000, +0.3819, +0.4986, +1.5103, +0.1718, +0.7768

α/π :

+0.0694, +0.6253, -0.1324, -0.1221, -0.1397, +0.2677, -0.4405, -0.1779, -0.1619, -0.1259
 -0.1578, -0.1802, -0.1826, -0.1465, -0.1474, -0.1877, -0.2098, -0.2238, -0.2025, +1.8100
 -0.8371, -0.3069, -0.6775, +0.2084, +0.2122, +0.1988, +0.1712, +0.1164, +0.1178, +0.1797
 +0.1887, +0.1933, +0.1669, +0.1046, +1.5286, +0.6546, +0.1825, +0.1317, +0.1728, +0.1617
 +0.1207, +0.1438, +0.2295, +0.1035, +0.3893, +1.3851, -0.5672, -0.0813, -0.1348, -0.7316
 -0.1805, +0.4700, +0.3483, +0.1291, -0.1354, +0.0785, +0.0607, +0.8060

$N = 9$ ($L = 63$, $T = 52.2657$, $T/N = 5.81$):

θ/π :

+0.0000, +0.0000, +0.0304, +0.3022, +0.3593, +0.5818, +0.2648, +0.0008, +0.1370, +0.1452
 +0.2035, +0.1083, +0.1590, +0.1087, +0.1013, +0.1346, +0.1923, +0.0925, +0.0552, +0.0508
 +0.0513, +0.0597, +0.1920, +0.0000, +0.0000, +0.0624, +0.0549, +0.0563, +0.0701, +0.1033
 +0.1751, +0.1409, +0.0917, +0.0793, +0.1144, +0.2591, +0.3963, +0.0000, +0.2932, +0.5230
 +0.4750, +0.4907, +0.5378, +0.6479, +0.6628, +1.1455, +0.8839, +0.6537, +0.2904, +0.0804
 +0.4127, +0.1069, +0.0846, +0.0000, +0.0000, +0.0005, +0.3421, +0.1561, +0.4818, +2.1109
 +0.1741, +0.4577, +0.6902

α/π :

+0.1762, +0.1523, +0.3015, -0.0565, -0.2891, -0.0945, +0.2492, -0.4572, -0.1883, -0.1799
 -0.1667, -0.0587, -0.1574, -0.1582, -0.1567, -0.1302, -0.1494, -0.1897, -0.2095, -0.2203
 -0.1933, +1.8395, -0.8335, -0.3037, -0.6717, +0.2143, +0.2185, +0.2051, +0.1741, +0.1114
 +0.1005, +0.1874, +0.2005, +0.2021, +0.1575, +0.0650, +1.5701, +0.6492, +0.2728, +0.2617
 +0.2898, +0.3958, -0.1752, -0.2972, +0.2261, +0.2285, +0.1079, +0.3884, +1.3839, -0.5681
 -0.0822, -0.1351, -0.7310, -0.1800, +0.4704, +0.3487, +0.1308, -0.0693, -0.1526, +0.5584
 -0.0194, +0.2010, +0.4299

$N = 10$ ($L = 70$, $T = 56.9380$, $T/N = 5.69$):

θ/π :

+0.1358, +0.3543, +0.0984, +0.0945, +0.3255, +0.1237, +0.3120, +0.0948, +0.3849, +0.1498
 +0.1470, +0.2052, +0.4987, +0.3172, +0.1977, +0.1783, +0.2411, +0.3573, +0.3690, +0.0600
 +0.1512, +0.1336, +0.1012, +0.3796, +0.7192, +0.2951, +0.5020, +0.5134, +0.2327, +0.2327

+0.4694, +0.2657, +0.2657, +0.4081, +0.1768, +0.1766, +0.1710, +0.1710, +0.3540, +0.2229
+0.2229, +0.1944, +0.1283, +0.2601, +0.1286, +0.1383, +0.3175, +0.1546, +0.1553, +0.5084
+0.1178, +0.1513, +0.2446, +0.2434, +0.2136, +0.1608, +0.1597, +0.1411, +0.1369, +0.1890
+0.1942, +0.1804, +0.1805, +0.6911, +0.5964, +0.5314, +0.2815, +0.4047, +0.3070, +0.2033

α/π :

+0.0871, -2.3822, -0.4261, -0.3905, +0.1951, -1.8188, -0.0455, -0.0635, +0.1969, +0.2118
+0.1933, +0.1485, +0.1669, +0.1523, +0.0666, +0.1360, -1.8310, -0.0122, +0.0998, +0.1494
+0.6600, +0.0904, +0.0469, +1.5504, -0.4001, -2.4494, -0.0304, +1.8833, -0.0046, -0.0042
+2.1149, -0.0001, +0.1278, +0.7046, +0.2152, -0.0267, -0.0010, -0.7323, -0.2973, -0.0184
-0.4691, -0.1894, -0.1334, +0.0299, +0.0379, +0.1087, -0.0394, -0.0220, -0.0436, +0.0379
-0.0286, +0.0540, +0.0136, -0.0941, -0.5751, -0.1015, +0.0109, +0.2153, +0.5154, +0.2779
+0.2165, +0.0381, -1.5422, -0.3238, +0.0341, -0.0731, -0.0314, +0.0551, +0.0354, +3.9414

$N = 11$ ($L = 75$, $T = 66.9068$, $T/N = 6.08$):

θ/π :

+0.1744, +0.1464, +0.1982, +0.1755, +0.2388, +0.1286, +0.1291, +0.2395, +0.2361, +0.2304
+0.2262, +0.2410, +0.2421, +0.2421, +0.2420, +0.2412, +0.2393, +0.2182, +0.2190, +0.2180
+0.2184, +0.2318, +0.2270, +0.4585, +0.2575, +0.2577, +0.2522, +0.1291, +0.1332, +0.2143
+0.2184, +0.3777, +0.4570, +0.2788, +0.2773, +0.4713, +0.4719, +0.2522, +0.2440, +0.4742
+0.2206, +0.2207, +0.4460, +0.4643, +0.4667, +0.8783, +0.4734, +0.2444, +0.1302, +0.1299
+0.9348, +0.2215, +0.2163, +0.2229, +0.2218, +0.2419, +0.2423, +0.1238, +0.1238, +0.2422
+0.2383, +0.2389, +0.1228, +0.1217, +0.2381, +0.4525, +0.2243, +0.2243, +0.4753, +0.2465
+0.2460, +0.4875, +0.4837, +0.4448, +0.4579

α/π :

+0.1846, +0.1104, -0.0754, -0.5387, -0.0476, -0.0036, -0.0259, -0.0745, +1.3112, -0.1231
-0.0823, -0.0103, -2.2654, -0.0055, -0.0104, -0.0229, -0.5519, -0.0247, +0.0570, +0.0096
+0.7686, +0.0193, +0.0729, +2.1638, -0.0058, -0.0345, -0.0874, -0.0274, -0.5983, -0.0494
+0.1204, -1.2137, +0.1149, -0.0129, +0.1086, +0.0071, -0.0195, +0.0177, +0.0191, -0.3737
-0.0038, +0.0284, +0.9608, +0.0248, +0.3007, +1.5426, -0.0359, -0.0104, +0.0074, -0.2106
+1.1467, -0.0555, -0.0760, +0.0225, +0.3595, +0.0086, -0.0027, +0.0004, +0.0046, +0.2144
+0.0102, -0.0128, -0.0192, -0.0131, -0.8614, -0.0259, +0.0046, +0.5237, +0.0147, +0.0016
+0.1511, +0.0299, -0.0403, -0.1559, -0.7436

$N = 12$ ($L = 78$, $T = 67.6302$, $T/N = 5.64$):

θ/π :

+0.8692, +0.2910, +0.2919, +0.2039, +0.3135, +0.4751, +0.4865, +0.4039, +0.3492, +0.4202
+0.2549, +0.2549, +0.1647, +0.1523, +0.3966, +0.2337, +0.2225, +0.2026, +0.1282, +0.1061
+0.0462, +0.1549, +0.1544, +0.5043, +0.2855, +0.2922, +0.2665, +0.2581, +0.1380, +0.1377
+0.2664, +0.4061, +0.3103, +0.2207, +0.2356, +0.2513, +0.2525, +0.3407, +0.3126, +0.3543
+0.5249, +0.2197, +0.2046, +0.4522, +0.4173, +0.6571, +0.1524, +0.1317, +0.1323, +0.4211
+0.2155, +0.4893, +0.2924, +0.1527, +0.1557, +0.4654, +0.1543, +0.1210, +0.3056, +0.0843
+0.0990, +0.1496, +0.3486, +0.2109, +0.2058, +0.4936, +0.1369, +0.1370, +0.6767, +0.1555
+0.1320, +0.3215, +0.1610, +0.1679, +0.3010, +0.3099, +0.1547, +0.2076

α/π :

+0.1806, +0.0057, -0.1432, -0.0917, -0.0607, -0.0193, +0.2837, -0.1281, -0.7897, -0.0690
+0.0050, -0.0860, -0.0791, -0.0835, +0.2380, -0.0150, +0.0074, +0.1837, -1.2634, +0.1764
+0.0863, -0.0074, -0.4093, -0.0025, -0.0688, -0.3214, -0.0540, -0.0273, +0.0087, +0.0201
-1.6335, -0.2273, +1.3088, -0.1371, -0.0401, -0.0086, -0.2964, +0.0388, -1.5772, +0.1500

-1.7774, +0.2213, +0.1500, +0.0459, +0.0770, +0.1364, +0.2307, +0.3011, +0.1869, -0.3178
-0.3239, -0.1548, -0.3085, -0.1849, -0.2542, -0.2897, -0.3971, -0.3149, +0.2601, +0.3061
-1.6330, +0.2379, +0.2108, +0.2357, +0.1940, -0.0447, -0.0208, -0.1105, +4.3610, +0.2697
+0.2907, -0.0754, +1.7694, -0.1580, -0.0510, -0.2320, -0.3778, -2.3293